SML (Introduction to functional analysis)
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This report shows the proof of the property on ordering the partial derivatives in any convenient order.:

If $f \in C^\infty(\mathbb{R}^n)$ and for any $j, k \in 1, 2, ..., n$ one has

$$\partial_j[\partial_k f] = \partial_k[\partial_j f]$$

To prove this statement, we have to accept one result: the mean value theorem. This theorem's statement is here:

Suppose f is a function which satisfies both of the following.
1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)

Then there is a number $c \in (a, b)$ such that $f(b) - f(a) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}(b-a)$

This theorem is rewritten as

For any $n \in \mathbb{N}$, when $f \in C^\infty(\mathbb{R}^n)$, and for any X $\in \mathbb{R}^n$, the following equation is true with given closed set $[[a_1, b_1]..., [a_j, b_j], ..., [a_n, b_n]]$:

$$f(X + hE_j) - f(X) = \partial_j f(P)\,(b_j - a_j) \; with \; Pr_j(P) \in (a_j, b_j)$$

Remark: Pr_j is a function $\mathbb{R}^n \to \mathbb{R}$, $(x_1, x_2, ..., x_j, ..., x_n)^t \mapsto x_j$.
It's time to go back to topic.
Obviously, to prove the property, all we have to do is prove the following statement (this statement is equal to the case: $n = 2$):

If $f \in C^\infty(\mathbb{R}^2)$ and we have

$$\partial_x[\partial_y f(x, y)] = \partial_y[\partial_x f(x, y)]$$

Therefore, we will see why the above statement is true.
We set $\varphi(x) = f(x, y + \Delta y) - f(x, y)$. As $f \in C^\infty(\mathbb{R}^2)$ and $C^\infty(\mathbb{R}^2) \subset C^0(\mathbb{R}^2)$, we can use mean value theorem for this φ, which implies
there exists $\theta_1 \in (0, 1)$ $s.t.$

$$\varphi(x + \Delta x) - \varphi(x) = \varphi'(x + \theta_1\Delta x)\,\Delta x = [\partial_x f(x + \theta_1\Delta x, y + \Delta y) - \partial_x f(x + \theta_1\Delta x, y)]\Delta x.$$

Note the function in the above bracket is a function of y. Then we can use mean value theorem again on this function:

$$[\partial_x f(x + \theta_1\Delta x, y + \Delta y) - \partial_x f(x + \theta_1\Delta x, y)]\Delta x = \partial_y\partial_x f(x + \theta_1\Delta x, y + \theta_2\Delta y)\Delta x\Delta y$$

with $\theta_2 \in (0, 1)$.
On the other hand, if we set

$$\partial_y\partial_x f(x + \theta_1\Delta x, y + \theta_2\Delta y) = \partial_y\partial_x f(x, y) + \varepsilon(\Delta x, \Delta y)$$

,because $\partial_y\partial_x f$ is continuous ($\because f \in C^\infty(\mathbb{R}^2)$), when $\Delta x, \Delta y \to 0$, $\varepsilon(\Delta x, \Delta y) \to 0$. Also, we can conclude

$$\varphi(x + \Delta x) - \varphi(x) = \{\partial_y\partial_x f(x, y) + \varepsilon(\Delta x, \Delta y)\}\Delta x\Delta y$$

Dividing this equation by Δy , and, after that, if $\Delta y \to 0$, we can get this equation:

$$\{\partial_y \partial_x f(x,y) + \lim_{\Delta y \to 0} \varepsilon(\Delta x, \Delta y)\}\Delta x = \lim_{\Delta y \to 0}\{\frac{\varphi(x+\Delta x)}{\Delta y} - \frac{\varphi(x)}{\Delta y}\} = \partial_y f(x+\Delta x) - \partial_y f(x)$$

Therefore, we can conclude

$$\partial_y \partial_x f(x,y) = \frac{\partial_y f(x+\Delta x) - \partial_y f(x)}{\Delta x} - \lim_{\Delta y \to 0} \varepsilon(\Delta x, \Delta y)$$

Considering $\Delta x \to 0$, we finally get

$$\partial_y \partial_x f(x,y) = \partial_x \partial_y f(x,y) \quad \square$$