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Exercise 1.2.5

Check that $Pv\frac{1}{x}$ is a distribution.

$$Pv\frac{1}{x}(f) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x} f(x)\, dx$$

① $$Pv\frac{1}{x}(f+\lambda h) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x}\{f(x)+\lambda h(x)\}\, dx$$

$$= \lim_{\varepsilon \searrow 0}\left\{\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x} f(x)\, dx + \lambda \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x} h(x)\, dx\right\}$$

$$= -\int_{\mathbb{R}} \ln(|x|) f'(x)\, dx - \lambda \int_{\mathbb{R}} \ln(|x|) h(x)\, dx$$

$$= Pv\frac{1}{x}(f) + \lambda Pv\frac{1}{x}(g)$$

$\boxtimes$

② When $\|f_n - f_\infty\| \xrightarrow{n\to\infty} 0$ in $D(\mathbb{R})$,

then $\|T(f_n) - T(f_\infty)\| \xrightarrow{n\to\infty} 0$

$\operatorname{supp} f_n \subset [-r, r] \quad \forall n \in \mathbb{N}$

$$\left|Pv\frac{1}{x}(f_n) - Pv\frac{1}{x}(f_\infty)\right|$$

$$= \left|-\int_{\mathbb{R}} \ln(|x|) f_n'(x)\, dx - \left\{-\int_{\mathbb{R}} \ln(|x|) f_\infty'(x)\, dx\right\}\right|$$

$$= \left|\int_{-r}^{r} \ln(|x|)\{f_n'(x) - f_\infty'(x)\}\, dx\right|$$

$$\leqq \int_{-r}^{r} |\ln(|x|)|\, dx\, \|f_n' - f_\infty'\|_\infty$$

$$= \int_{-r}^{r} |\ln(|x|)|\, dx\, \|f_n' - f_\infty'\|_\infty$$

$^{\forall}\varepsilon > 0$,

choose ε' satisfying $\left|\int_{-r}^{r} \ln(|x|)\,dx\right| \varepsilon' < \varepsilon$

$\lim_{n\to\infty} \|f_n - f_\infty\|_\infty = 0$

$^{\forall}\varepsilon' > 0$, $^{\exists}N \in \mathbb{N}$ s.t. $^{\forall}n \geqq N$, $\|f_n - f_\infty\|_\infty < \varepsilon'$

For $n \geqq N$,

$$\left|\int_{-r}^{r} \ln(|x|)\,dx\right| \|f_n - f_\infty\|_\infty < \left|\int_{-r}^{r} \ln(|x|)dx\right| \varepsilon' < \varepsilon \quad \blacksquare$$

Compute that exact expression for $O(\varepsilon|\ln(\varepsilon)|)$

$$\mathrm{Pv}\frac{1}{x}(f) = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x} f(x)\,dx$$

$$= \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} \frac{1}{x} f(x)\,dx + \int_{\varepsilon}^{\infty} \frac{1}{x} f(x)\,dx\right)$$

$$= \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} \ln(|x|)' f(x)\,dx + \int_{\varepsilon}^{\infty} \ln(|x|)' f(x)\,dx\right)$$

$$= \lim_{\varepsilon\searrow 0}\left\{\ln(|x|)f(x)\Big[_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon}\ln(|x|)f'(x)\,dx + \ln(|x|)f(x)\Big|_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty}\ln(|x|)f'(x)\,dx\right\}$$

$$= -\lim_{\varepsilon\searrow 0}\left\{\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\ln(|x|)f'(x)\,dx - \ln(|\varepsilon|)f(-\varepsilon) + \ln(|\infty|)f(-\infty) - \ln(|\infty|)f(\infty) + \ln(|\varepsilon|)f(\varepsilon)\right\}$$

$$= -\lim_{\varepsilon\searrow 0}\left[\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\ln(|x|)f'(x)\,dx + \ln(\varepsilon)\{f(\varepsilon) - f(-\varepsilon)\} + \ln(\infty)\{f(-\infty) - f(\infty)\}\right]$$

Since f is a test function, $f(\pm\infty) = 0$

$$O(\varepsilon|\ln(\varepsilon)|) = \ln(\varepsilon)\{f(\varepsilon) - f(-\varepsilon)\}$$

$$= \varepsilon\ln(|\varepsilon|)\left\{\frac{f(\varepsilon) - f(0)}{\varepsilon} + \frac{f(-\varepsilon) - f(0)}{-\varepsilon}\right\} \quad //$$

$$\left(\lim_{\varepsilon \to 0} O(\varepsilon \ln(|\varepsilon|)) = 0 \cdot \{ f'(0) + f'(0) \} = 0 \right)$$