Exercise 1.1.9

Show that Th belongs to $D'(\mathbb{R}^n)$ by proving the conditions below,

1. $T(f_1 + \lambda f_2) = T(f_1) + \lambda T(f_2)$ for any $f_1, f_2 \in D(\mathbb{R}^n)$ and $\lambda \in \mathbb{K}$
2. $T(f_n) \xrightarrow{n\to\infty} T(f)$ when $D(\mathbb{R}^n) \ni f_n \xrightarrow{n\to\infty} f \in D(\mathbb{R}^n)$

$T_h(f) = \int_{\mathbb{R}^n} h(x) f(x)\, dx$

1. Set $g \in D(\mathbb{R}^n)$ and $\lambda \in \mathbb{K}$.

$$\begin{aligned} T_h(f + \lambda g) &= \int h(f + \lambda g)\, dx \\ &= \int hf\, dx + \lambda \int hg\, dx \\ &= T_h(f) + \lambda T_h(g) \end{aligned}$$

Therefore, condition 1 is proven.
($\because f \in D(\mathbb{R}^n)$, $g \in D(\mathbb{R}^n)$, $\lambda \in \mathbb{K}$)

2. When $\lim_{n\to\infty} f_n = f_\infty$,
$\exists r$, $\operatorname{supp} f_n \subset B_r(0)$ $\forall n$

$$\begin{aligned} |T_h(f_n) - T_h(f_\infty)| &= \left| \int_{B_r(0)} h(x) f_n(x)\, dx - \int_{B(r)} h(x) f_\infty(x)\, dx \right| \\ &= \left| \int_{B_r(0)} h(x) \{ f_n(x) - f_\infty(x) \}\, dx \right| \\ &\leqq \left| \int_{B_r(0)} h(x)\, dx\, \| f_n - f_\infty \|_\infty \right| \\ &= \left| \int_{B_r(0)} h(x)\, dx \right| \| f_n - f_\infty \|_\infty \end{aligned}$$

(i) when $\left|\int_{B_r(0)} h(x)\,dx\right| \neq 0$,

$^\forall \varepsilon > 0$,

choose ε' satisfying $\left|\int_{B_r(0)} h(x)\,dx\right| \varepsilon' < \varepsilon$

$\lim_{n\to\infty} \| f_n - f_\infty \|_\infty = 0$

$^\forall \varepsilon' > 0$, $^\exists N \in \mathbb{R}$ s.t. $^\forall n \geq N$, $\| f_n - f_\infty \|_\infty < \varepsilon'$

For $n \geq N$,

$$\left|\int_{B_r(0)} h(x)\,dx\right| \| f_n - f_\infty \|_\infty < \left|\int_{B_r(0)} h(x)\,dx\right| \varepsilon' < \varepsilon$$

$\blacksquare$

(ii) when $\left|\int_{B_r(0)} h(x)\,dx\right| = 0$,

$$|T_h(f_n) - T_h(f_\infty)| = \left|\int_{B_r(0)} h(x)\,dx\right| \| f_n - f_\infty \|_\infty$$

$$= 0 < \varepsilon$$

$\blacksquare$