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Exercise 3.2.6

$$\langle f, M_\varphi g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} \, [M_\varphi g](x) \, dx$$

$$= \int_{\mathbb{R}^n} \overline{f(x)} \, \varphi(x) \, g(x) \, dx$$

Thus

$$\langle M_\varphi^* f, g \rangle = \langle f, M_\varphi g \rangle$$

$$= \int_{\mathbb{R}^n} \overline{f(x)} \, \varphi(x) \, g(x) \, dx$$

$$= \int_{\mathbb{R}^n} \overline{f(x) \, \overline{\varphi(x)}} \, g(x) \, dx$$

$$= \int_{\mathbb{R}^n} \overline{[M_{\overline{\varphi}} f]} (x) \, g(x) \, dx$$

$$= \langle M_{\overline{\varphi}} f, g \rangle$$

$\Rightarrow M_\varphi^* f = M_{\overline{\varphi}} f$ Since f is arbitrary,

$$M_\varphi^* = M_{\overline{\varphi}}$$

Exercise 3.3.3

If M_φ is a self-adjoint operator,

$M_\varphi = M_\varphi^*$

$\Rightarrow M_\varphi f = M_\varphi^* f$ ($\because f$ is arbitrary)

From exercise 3.2.6, $M_\varphi^* f = M_{\bar\varphi} f$

$\Rightarrow \varphi(x) f(x) = \bar\varphi(x) f(x)$ for a.e. x

$\Rightarrow \varphi = \bar\varphi$

$\Rightarrow \varphi$ is real

Next, if φ is real,

$\varphi = \bar\varphi$

$\Rightarrow \varphi(x) f(x) = \bar\varphi(x) f(x)$ for a.e. x.

From definition of $M_\varphi f$ that $[M_\varphi f](x) = \varphi(x) f(x)$

$\Rightarrow M_\varphi f = M_{\bar\varphi} f$

However, from exercise 3.2.6, $M_{\bar\varphi} f = M_\varphi^* f$, so

$M_\varphi f = M_\varphi^* f$

M_φ is a self-adjoint operator.

M_φ is a self-adjoint operator if and only if φ is a real valued function.