Properties of Fourier Transform

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Exercise 1.5.1

Prove some of the properties of Fourier Transform which is given as follows. For $f, g \in L^1(\mathbb{R}^n)$:

1. *$\mathcal{F}$ is a linear map on $L^1(\mathbb{R}^n)$,*
2. $\left|\hat{f}(\xi)\right| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(X)| dX$,
3. *$\hat{f}$ belongs to $C_0(\mathbb{R}^n)$, meaning that $\hat{f}$ is a continuous function on $\mathbb{R}^n$ satisfying $\lim_{||\xi|| \to \infty} \hat{f}(\xi) = 0$,*
4. *$\mathcal{F}(f * g) = \widehat{f * g} = \hat{f}\hat{g}$, where the convolution of f and g is defined by*
$$[f * g](X) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(X - Y) g(Y) dY,$$
5. *If $\partial_j f$ exists and belong to $L^1(\mathbb{R}^n)$, then*
$$[\mathcal{F}(-i\partial_j f)](\xi) \equiv [\widehat{-i\partial_j f}](\xi) = \xi_j \hat{f}(\xi).$$

Proof:

1. From the definition of Fourier transform of functions $f, g \in L^1(\mathbb{R}^n)$, one has
$$\begin{aligned}
[\mathcal{F}(af + g)](\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot X} \{af(X) + g(X)\} dX \\
&= a \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot X} f(X) dX + \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot X} g(X) dX \\
&= a[\mathcal{F}f](\xi) + [\mathcal{F}g](\xi)
\end{aligned}$$
for some constant $a \in \mathbb{R}$. Hence, $\mathcal{F}$ is a linear map on $L^1(\mathbb{R}^n)$ $\square$

2. For $f \in L^1(\mathbb{R}^n)$, let us express $\left|\hat{f}(\xi)\right|$

$$\begin{aligned}\left|\hat{f}(\xi)\right| &= \left|\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X)dX\right| \\ &= \frac{1}{(2\pi)^{n/2}}\left|\int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X)dX\right| \\ &\le \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\left|e^{-i\xi\cdot X}\right| |f(X)|\, dX \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} |f(X)|\, dX\end{aligned}$$

where we have used the equality $\left|e^{-i\xi\cdot X}\right| = 1$ which holds true whatever the value of $\xi \cdot X$ is. Hence, one has

$$\left|\hat{f}(\xi)\right| \le \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} |f(X)|\, dX.$$

□

3. We show only the continuity of $\hat{f}$. Consider $\eta \in \mathbb{R}^n$ such that one has

$$\begin{aligned}\hat{f}(\xi+\eta) - \hat{f}(\xi) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\left(e^{-i(\xi+\eta)\cdot X} - e^{-i\xi\cdot X}\right) f(X)dX \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot X}\left(e^{-i\eta\cdot X} - 1\right) f(X)dX.\end{aligned}$$

For any $f \in L^1$ and $\varepsilon > 0$, we can find $r > 0$ such that

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n\setminus\mathcal{B}_r(0)} |f(X)|dX < \frac{\varepsilon}{4}$$

which comes from the definition of L^1. For finite n, let $\delta = \frac{(2\pi)^{n/2}\varepsilon}{2r||f||_{L^1}}$ and for $x \in \mathcal{B}_r(0)$ and $|\eta| < \delta$, one has

$$|e^{-i\eta\cdot X} - 1| \le |i\eta\cdot X| \le |\eta||X| \le \frac{(2\pi)^{n/2}\varepsilon}{2||f||_{L^1}}$$

where we have used Cauchy-Schwarz inequality for the inner product.

Claim: For $x \in \mathbb{R}$, the inequality $|e^{-ix} - 1| \le |x|$ holds true.
Proof: Let us express $|e^{-ix} - 1|$ explicitly

$$|e^{-ix} - 1| = |(\cos x - 1) - i\sin x| = \sqrt{2 - 2\cos x} = 2\left|\sin\left(\frac{x}{2}\right)\right| \le 2\left|\frac{x}{2}\right| = |x|.$$

We have proved the claim.

Observe that for all $\eta \cdot X$, one has

$$|e^{-i\eta\cdot X} - 1| \le 2$$

Then, for $|\eta| < \delta$, one has

$$
\begin{aligned}
\left|\hat{f}(\xi+\eta) - \hat{f}(\xi)\right| &\le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left|e^{-i\eta\cdot X} - 1\right| |f(X)| dX \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n \setminus \mathcal{B}_r(0)} \left|e^{-i\eta\cdot X} - 1\right| |f(X)| dX + \frac{1}{(2\pi)^{n/2}} \int_{\mathcal{B}_r(0)} \left|e^{-i\eta\cdot X} - 1\right| |f(X)| dX \\
&\le \frac{\varepsilon}{2} + \frac{1}{(2\pi)^{n/2}} ||f||_{L^1} \frac{(2\pi)^{n/2}\varepsilon}{2||f||_{L^1}} \\
&= \varepsilon.
\end{aligned}
$$

We have proved that $\hat{f}$ is continuous.

4. From the definition, the Fourier transform of a convolution of two functions $f, g \in L^1(\mathbb{R}^n)$ can be written as

$$
\mathcal{F}(f * g) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X - Y) g(Y) dY dX
$$

Let us define a new variable $Z = X - Y$ such that $X = Z + Y$ and one has

$$
\begin{aligned}
\mathcal{F}(f * g) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot(Z+Y)} f(Z) g(Y) dZ dY \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot Z} f(Z) dZ \cdot \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot Y} g(Y) dY \\
&= \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}
$$

where we have used the distributive property of the scalar product in $\mathbb{R}^n$. □

5. Let us fix $j = n$, then we have the following expression

$$
\begin{aligned}
\xi_n \hat{f}(\xi) &= \xi_n \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X) dX \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_n e^{-i\xi\cdot X} f(X) dX \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} i \left(\frac{d}{dx_n} e^{-i\xi\cdot X}\right) f(X) dX \\
&= \frac{1}{(2\pi)^n} i \int_{\mathbb{R}^{n-1}} dx_1 \dots dx_{n-1} \int_{\mathbb{R}} dx_n \left(\frac{d}{dx_n} e^{-i\xi\cdot X}\right) f(x_1, \dots, x_n) \\
&= \frac{1}{(2\pi)^n} i \int_{\mathbb{R}^{n-1}} dx_1 \dots dx_{n-1} \left[\lim_{N\to\infty} e^{-i\xi\cdot X} f(x_1, \dots, x_n)\Big|_{x_n=-N}^{x_n=N}\right] \\
&\quad - \frac{1}{(2\pi)^n} i \int_{\mathbb{R}^n} e^{-i\xi\cdot X} (\partial_n f)(X) dX
\end{aligned}
$$

We want to show that the limit term is equal to zero. Consider the mapping $\mathbb{R}_+ \ni m \mapsto g(m) := \int_{\mathbb{R}^{n-1}} dx_1 \dots dx_{n-1} e^{-i\sum_{j=1}^{n-1} \xi_j x_j} e^{-i\xi_n m} f(x_1, \dots, x_{n-1}, m) \in \mathbb{C}$. The map is a Cauchy sequence since any $\epsilon > 0$, $\exists m > 0$ such that for any $m_1 > m_2 > m$, one has

$$|g(m_1) - g(m_2)| = \left| \int_{\mathbb{R}^{n-1}} dx_1 \dots dx_{n-1} e^{-i \sum_{j=1}^{n-1} \xi_j x_j} \left(e^{-i\xi_n m_1} f(x_1, \dots, x_{n-1}, m_1) \right.\right.$$
$$\left.\left. - e^{-i\xi_n m_2} f(x_1, \dots, x_{n-1}, m_2) \right) \right|$$

Let us set $Y = (x_1, \dots, x_{n-1})$ and $\xi' = (\xi_1, \dots, \xi_{n-1})$ such that one has

$$\begin{aligned}
|g(m_1) - g(m_2)| &= \left| \int_{\mathbb{R}^{n-1}} dY e^{-i\xi' \cdot Y} \left(e^{-i\xi_n m_1} f(Y, m_1) - e^{-i\xi_n m_2} f(Y, m_2) \right) \right| \\
&= \left| \int_{\mathbb{R}^{n-1}} dY e^{-i\xi' \cdot Y} \left(\int_{m_2}^{m_1} \frac{d}{ds} \left(e^{-i\xi_n s} f(Y, s) \right) ds \right) \right| \\
&= \left| \int_{\mathbb{R}^{n-1}} dY e^{-i\xi' \cdot Y} \left(\int_{m_2}^{m_1} \left(-i\xi_n e^{-i\xi_n s} f(Y, s) + e^{-i\xi_n s} \partial_n f(Y, s) \right) ds \right) \right| \\
&\leq \int_{\mathbb{R}^{n-1}} dY \int_{m_2}^{m_1} \left(|\xi_n f(Y, s)| + |\partial_n f(Y, s)| \right) ds \\
&\leq \int_{\mathbb{R}^{n-1}} dY \int_{m_2}^{\infty} \left(|\xi_n f(Y, s)| + |\partial_n f(Y, s)| \right) ds \\
&\leq \varepsilon
\end{aligned}$$

the expression holds true for any m large enough since f and $\partial_n f$ are in $L^1(\mathbb{R}^n)$ by our assumption. Then, by the same argument, the map $\mathbb{R}_- \ni m \mapsto g(m) \in \mathbb{C}$ is also a Cauchy sequence. In $\mathbb{R}$, the Cauchy sequence converges, which means $\lim_{m \to \pm\infty} g(m)$ exists, which means that the 2 limit terms discussed previously exist. Let us show that $g \in L^1(\mathbb{R})$. Since $f \in L^1(\mathbb{R}^n)$, one has

$$\begin{aligned}
\int_{\mathbb{R}} |g(m)| dm &= \int_{\mathbb{R}} dm \left| \int_{\mathbb{R}^{n-1}} dY e^{-i\xi' \cdot Y} e^{-i\xi' \cdot m} f(Y, m) \right| \\
&\leq \int_{\mathbb{R}} dm \int_{\mathbb{R}^{n-1}} dY |f(Y, m)| \\
&= ||f||_{L^1(\mathbb{R}^n)}
\end{aligned}$$

Thus, $g \in L^1(\mathbb{R})$ and $\lim_{m \to \pm\infty} g(m)$ exist and this implies that $\lim_{m \to \pm\infty} g(m) = 0$, otherwise it cannot be $L^1(\mathbb{R})$ (Since the limit at $m \to \pm\infty$ is a constant and by the condition that $g \in L^1(\mathbb{R})$, namely the integral of $|g|$ over $\mathbb{R}$ is bounded, then the constant of that limit has to be equal to zero). Hence, for the expression that we had previously, for any $j = 1, 2, \dots, n$, one has

$$\begin{aligned}
\xi_j \hat{f}(\xi) &= -\frac{1}{(2\pi)^n} i \int_{\mathbb{R}^n} e^{-i\xi \cdot X} (\partial_j f)(X) dX \\
&= -i[\mathcal{F}(\partial_j f)](\xi) \\
&= [\mathcal{F}(-i\partial_j f)](\xi)
\end{aligned}$$

□