On properties of the Lebesgue outer measure

Some necessary definitions:

$I := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_j \leq x_j \leq b_j \ \forall j \in \{1, \dots, n\}\}$

with $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ is called closed box.

$v(I) := \prod_{j=1}^{n} (b_j - a_j) > 0$ is called volume of I

$S := \{I_j\}_j$ is a covering of Ω if $\Omega \subset \bigcup_j I_j$

$\sigma(S) := \sum_j v(I_j) \in (0, \infty]$ is called the volume of the covering.

$m^*(\Omega) := \inf \{\sigma(S) \mid S \text{ covering of } \Omega\}$ is called Lebesgue outer measure of Ω

most explicit form: $m^*(\Omega_1) = \inf \{ \sum_K \prod_{j=1}^{n} (b_{Kj} - a_{Kj}) \}$

Exercise 2.2.5: i) If $\Omega_1 \subset \Omega_2$, then $m^*(\Omega_1) \leq m^*(\Omega_2)$, $\Omega_i \subset \mathbb{R}^n$, $i \in \{1, 2\}$

Proof: Let S be a covering of Ω_2 by closed boxes I_K, $K \in \{1, \dots \ell\}$, ℓ number of closed boxes in $\mathbb{R}^n$

since $\Omega_1 \subset \Omega_2$, S is also a covering of Ω_1.

$\Rightarrow m^*(\Omega_1) = \inf \{\sigma(S) \mid S \text{ covering of } \Omega_1\} \leq \sigma(S)$ where $\sigma(S)$ already covers Ω_2

$\Rightarrow m^*(\Omega_2) = \inf \{\sigma(S) \mid S \text{ covering of } \Omega_2\} \leq \sigma(S)$

side remark: $\sigma(S)$ is one upper bound for $m^*(\Omega_{1/2})$

Also, $\{\sigma(S) \mid S \text{ covering of } \Omega_2\} \subseteq \{\sigma(S) \mid S \text{ covering of } \Omega_1\}$ because the set of possible coverings is larger for smaller sets Ω.

Taken into account the property of the infimum, that for an arbitrary set $A \subseteq B$ it holds, that $\inf\{A\} \leq \inf\{B\}$.

Hence, we get $m^*(\Omega_1) \leq m(\Omega_2)$.

ii) $m^*(\Omega_1 \cup \Omega_2) \leq m^*(\Omega_1) + m^*(\Omega_2)$

Proof: Let S, T coverings of A, B by closed intervals respectively, then $S \cup T$ forms a covering of $A \cup B$ by closed intervals

$\Rightarrow m^*(A \cup B) \leq \sigma(S \cup T) \leq \sigma(S) + \sigma(T)$

↳ since some closed boxes might overlap

Note: If $m^*(A)$ or $m^*(B)$ is infinite $m^*(A \cup B)$ is infinite and $\infty \leq \infty$ by convetion

Let $m^*(A)$, $m^*(B)$ be finite and $\mathbb{R} \ni \varepsilon > 0$.

Then $\exists\, S$ s.t. $m^*(A) \leq \sigma(S) \leq m^*(A) + \frac{\varepsilon}{2}$ and

$m^*(B) \leq \sigma(T) \leq m^*(B) + \frac{\varepsilon}{2}$ by closed intervals

$\Rightarrow m^*(A \cup B) \leq \sigma(S \cup T) \leq \sigma(S) + \sigma(T) \leq m^*(A) + m^*(B) + \varepsilon$

$\Rightarrow m^*(A \cup B) \leq m^*(A) + m^*(B)$ for an arbitrary ε

iii) $m^*(\bigcup_j \Omega_j) \leq \sum_j m^*(\Omega_j)$ for a finite or countable (infinte) family (of sets out of $\mathbb{R}^n$)

Proof: Additionally to the above assumptions of ii), let $n \in \mathbb{N} \cup \{\infty\}$ countable with a respective covering of A_n by S_n by closed intervals.

$\Rightarrow m^*(\bigcup_{i=1}^{n} A_n) \leq \sigma(\bigcup_{i=1}^{n} S_n) \leq \sum_{i=1}^{n} \sigma(S_n) \leq \sum_{i=1}^{n} (m^*(A_n) + \frac{\varepsilon}{2^n}) \overset{\text{split sum}}{=} \varepsilon + \sum_{i=1}^{n} m^*(A_n)$

and as before ε arbitrary. Hence $m^*(\bigcup_{i=1}^{n} A_n) \leq \sum_{i=1}^{n} m^*(A_n)$ holds.

Note: $\exists\, \Omega_1, \Omega_2$ s.t. $\Omega_1 \cap \Omega_2 = \emptyset \wedge m^*(\Omega_1 \cup \Omega_2) < m^*(\Omega_1) + m^*(\Omega_2)$

does not contradict ii) and iii)