On natural cubic splines as test function approximations
(Exercise 1.1.5: Exhibit other elements of $\mathcal{D}(\mathbb{R}^n)$. (without using convolution))

Def. Test function: A smooth $f: \mathbb{R}^n \to \mathbb{K}$ with bounded support is called a test function. In other words, if $f \in C^\infty(\mathbb{R}^n)$ and $supp(f) \in B_r(0)$ for $r \in \mathbb{R}$ large enough, then f is a test function. The set of all test functions is denoted by $\mathcal{D}(\mathbb{R}^n)$

Def. Natural Splines: On an intervall $[a,b]$, we denote $C^k[a,b]$ the space of all k-times continously differentiable functions. We call a function S a spline of degree k on $[a,b]$ if
i) $S \in C^{k-1}([a,b])$
ii) $a = t_0 \le t_1 \le \ldots \le t_{N+1} = b$ and

$$S(x) = \begin{cases} S_1(x), & t_0 \le x \le t_1 \\ S_2(x), & t_1 \le x \le t_2 \\ \vdots & \\ S_{N+1}(x), & t_N \le x \le t_{N+1} \end{cases}$$

where S_i is a polynomial of degree K and $N+1$ the number of subintervals.

Def. Cubic Spline: A spline of degree 3 is called a cubic spline and has the form

$$S(x) = \begin{cases} S_1(x) = a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3, & t_0 \le x \le t_1 \\ \vdots & \\ S_{N+1}(x) = a_{N0} + a_{N1}x + a_{N2}x^2 + a_{N3}x^3, & t_N \le x \le t_{N+1} \end{cases}$$

to enforce countinous differentiability, i.e. $S \in C^2([a,b])$, following conditions need to be met

$$\begin{aligned} S_i(x_i) &= S_{i+1}(x_i) \\ S_i'(x_i) &= S_{i+1}'(x_i) \quad \forall\, i = 1,\ldots,N \\ S_i''(x_i) &= S_{i+1}''(x_i) \end{aligned}$$

Remark: A cubic spline with N subintervalls has $4(N+1)$ degrees of freedom (DoF) i.e. the coefficients $(a_{i0}, a_{i1}, a_{i2}, a_{i3})_{i=1}^{N+1}$.

How to set conditions to approximate test functions?

$$\begin{aligned} a_{00} + a_{01}x + a_{02}x^2 + a_{03}x^3 &= S_0(x) \\ a_{01} + 2a_{02}x + 3a_{03}x^2 &= S_0'(x) \\ 2a_{02} + 6a_{03}x &= S_0''(x) \qquad (*) \\ 6a_{03} &= S_0^{(3)}(x) \\ 0 &= S_0^{(4)}(x) \\ &\vdots \end{aligned}$$

A test function not only requires C^{K-1} (see nat. splines (i)) but C^∞ which means for cubic splines:

$$\begin{aligned} S_i(x_i) &= S_{i+1}(x_i) \\ S_i'(x_i) &= S_{i+1}'(x_i) \quad \forall\, i = 1,\ldots,N \\ S_i''(x_i) &= S_{i+1}''(x_i) \quad \text{(where } x_i \text{ is a knot connecting two splines)} \qquad (**) \\ S_i^{(3)}(x_i) &= S_{i+1}^{(3)}(x_i) \end{aligned}$$

Claim: The only spline that statify the definition of a test function is the trivial spline where $a_{ij} = 0 \;\forall\, i,j$.

Proof: Recall, a test function has compact support $\Rightarrow S(t_0) = 0 = S(t_{N+1})$. Since, for a real interval $[t_0, t_{N+1}]$, $t_0 \ne t_{N+1}$ we cannot assume $S = 0$ $\Rightarrow$ respective $a_{0j} = 0, j \in \{0,\ldots,K\}$. From $(*)$ and $(**)$, we can see that $2(K+1)$ DoF are necessary at the boundary points t_0, t_{N+1}. Furthermore, to connect $N+1$ subintervals, N knots are critical points to observe under diffeutiation. All non critical points, $x \notin sup(S_i) \wedge sup(S_{i+1})$, are strictly polynomial and naturally in $C^\infty((t_i, t_{i+1}))\; \forall i$. For the remaining critical points at each of the N knots, additional $N(K+1)$ DoF are necessary. In sum, $(N+2)(K+1)$ DoF are required to get a test function. A spline of degree K has $K(N+1)$ DoF. Hence, $(K+1)(N+2) = K(N+1) + (N+2) + K > K(N+1)$ and the only spline which is a test function, is the trivial spline

$\square$