Proof : $S(\mathbb{R}^n) \in L^1(\mathbb{R}^n)$

① $n=1$

let $f \in S(\mathbb{R})$, then $\|x^2 f(x)\| < \infty$

$\exists k>0, x_1>0, \forall |x|>x_1$. s.t. $x^2|f(x)|<k$

then

$$\int_{-\infty}^{+\infty} |f(x)|dx = \int_{\mathbb{R}\setminus[-x_1,x_1]} |f(x)|dx + \int_{-x_1}^{x_1} |f(x)|dx < \int_{\mathbb{R}\setminus[-x_1,x_1]} \frac{k}{x^2}dx + \int_{-x_1}^{x_1}|f(x)|dx.$$

It's obvious that $\int_{\mathbb{R}\setminus[-x_1,x_1]} \frac{k}{x^2}dx<\infty$, $\int_{-x_1}^{x_1} |f(x)|dx<\infty$ (f is continuous).

so $\int_{-\infty}^{+\infty}|f(x)|dx<\infty$

② n is general

for $f \in S(\mathbb{R}^n)$, we get $\lim_{\|X\|\to\infty} \|X\|^{n+1}|f(X)| < \infty$

then $\exists K>0, r>0$, st. $\forall \|X\|>r$, $\|X\|^{n+1}|f(X)|<K$

so $\int_{\mathbb{R}^n\setminus B_r(0)} |f(X)|dX < \int_{\mathbb{R}^n\setminus B_r(0)} \frac{K}{\|X\|^{n+1}}dX$

Turn to spherical coordinates for n dimension

$$\int_{\mathbb{R}^n\setminus B_r(0)} \frac{K}{\|X\|^{n+1}}dX = c\cdot\int_r^\infty \frac{K}{\rho^{n+1}}\cdot\rho^{n-1}d\rho = c\cdot\int_r^\infty \frac{K}{\rho^2}d\rho<\infty. \ (c\in\mathbb{R}, c<\infty)$$

ρ^{n-1} comes from Jacobian, actually

$$\text{let.}\begin{cases} x_n = \rho\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1} \\ x_{n-1} = \rho\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ x_{n-2} = \rho\sin\theta_1\cdots\cos\theta_{n-2} \\ \vdots \\ x_2 = \rho\sin\theta_1\cos\theta_2 \\ x_1 = \rho\cos\theta_1 \end{cases} \Rightarrow \begin{cases} \sum_{i=1}^n x_i^2 = \rho^2 \Rightarrow \rho = \|X\| \\ J = \frac{\partial(x_1,\cdots,x_n)}{\partial(\rho,\theta_1,\cdots\theta_{n-1})} = \rho^{n-1}f(\theta_1,\cdots,\theta_{n-1}) \end{cases}$$

then $\int_{\mathbb{R}^n}|f(X)|dX = \int_{B_r(0)}|f(X)|dX + \int_{\mathbb{R}^n\setminus B_r(0)}|f(X)|dX < \int_{B_r(0)}|f(X)|dX + \int_{\mathbb{R}^n\setminus B_r(0)}\frac{K}{\|X\|^{n+1}}dX < \infty$

so. if $f\in S(\mathbb{R}^n)$, then $f\in L^1(\mathbb{R}^n) \Rightarrow S(\mathbb{R}^n)\in L^1(\mathbb{R}^n)$. □

The idea of $\|X\|^{n+1}$ was inspired by Prof. Richard.