Definition 2.2.7 (Lebesgue measurability, Lebesgue measure)**.** *A set* $\Omega \subset \mathbb{R}^n$ *is* Lebesgue measurable *if for any* $\varepsilon > 0$ *there exists an open set* Λ *with* $\Omega \subset \Lambda$ *and such that*

$$m^*(\Lambda \setminus \Omega) \leq \varepsilon.$$

For any Lebesgue measurable set Ω*, we define* $m(\Omega) := m^*(\Omega)$ *and call it the* Lebesgue measure *of* Ω*.*

1. If Ω is an open set, then Ω is Lebesgue measurable

Proof: $m^*(\Omega \setminus \Omega) = m^*(\emptyset) = 0 < \varepsilon$ for every $\varepsilon > 0$

2. If $m^*(\Omega) = 0$, then Ω is Lebesgue measurable

Proof:

Use

Theorem 2.2.6. *Let* $\Omega \subset \mathbb{R}^n$ *with* $m^*(\Omega) < \infty$*. Then for any* $\varepsilon > 0$ *there exists an open set* Λ *with* $\Omega \subset \Lambda$ *and such that*

$$m^*(\Lambda) \leq m^*(\Omega) + \varepsilon.$$

If $m^*(\Omega) = \infty$*, the statement holds with* $\Lambda = \mathbb{R}^n$*.*

Choose open set Λ such that $\Omega \subset \Lambda$

$\Rightarrow m^*(\Lambda) < m^*(\Omega) + \varepsilon = \varepsilon$

Since $\Lambda \setminus \Omega \subset \Lambda \Leftrightarrow m^*(\Lambda \setminus \Omega) < m^*(\Lambda)$

$$m^*(\Lambda \setminus \Omega) \leq m^*(\Lambda) < \varepsilon$$

$\Leftrightarrow \Omega$ is Lebesgue measurable

3. If $\Omega := \bigcup_j \Omega_j$ is a finite or countable union of Lebesgue measurable sets, then Ω is Lebesgue measurable and $m(\Omega) \leq \sum_j m(\Omega_j)$

Proof: For each Ω_j there exist an open set Λ_j, with $\Omega_j \subset \Lambda_j$

$m^*(\Lambda_j \setminus \Omega_j) < \frac{\varepsilon}{2^j}$, since Ω_j is Lebesgue measurable

Consider now $\Lambda = \bigcup_j \Lambda_j$, $\Omega \subset \Lambda$

$$m^*(\Lambda \setminus \Omega) = m^*(\bigcup_j \Lambda_j \setminus \bigcup_j \Omega_j)$$
$$\leq m^*(\bigcup_j (\Lambda_j \setminus \Omega_j))$$
$$\leq \sum_j m^*(\Lambda_j \setminus \Omega_j)$$
$$< \sum_j \frac{\varepsilon}{2^j} \leq \varepsilon$$

$\Rightarrow \Omega$ is measurable and $m(\Omega) \leq \sum_j m(\Omega_j)$

4. If $\Omega = \bigcap_j \Omega_j$ is a finite or countable intersection of Lebesgue measurable sets, then Ω is Lebesgue measurable

Proof: $\Omega^c = (\bigcap_j \Omega_j)^c = \bigcup_j \Omega_j^c$

Use **Theorem 2.2.10.** *If Ω is a Lebesgue measurable set, then its complement Ω^C is also a Lebesgue measurable set.*

Due to 3., Ω^c is Lebesgue measurable, so Ω is also Lebesgue measurable

5. Any closed set is Lebesgue measurable, in particular any closed box I is Lebesgue measurable, with $m(I) = v(I)$

Proof: First: Any closed set is Lebesgue measurable

The complement of a closed set is an open set. Since every open set in $\mathbb{R}^n$ is Lebesgue measurable and the complement of an Lebesgue measurable set is also mesurable, every closed set in $\mathbb{R}^n$ is Lebesgue measurable.

$I = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1 \le x_1 \le b_1, \dots, a_n \le x_n \le b_n\}$

Consider minimal covering of I by rectangular division of I.

Let $a_i = p_{0i} < p_{1i} < \dots < p_{mi} = b_i$. For $j_i = 1, 2, \dots, m$

$I_{j_1 \cdots j_n} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid p_{j_1 - 1} \le x_1 \le p_{j_1}, \dots, p_{j_n - 1} \le x_n \le p_{j_n}\}$

Volume: $v(I) = \prod_{i=1}^{n} (b_i - a_i)$

$$
\begin{aligned}
\sum_{j_1}^{m} \dots \sum_{j_n}^{m} v(I_{j_1 \cdots j_n}) &= \sum_{j_1}^{m} \dots \sum_{j_n}^{m} \prod_i^n (p_{ji} - p_{ji-1}) \\
&= \sum_{j_1}^{m} (p_{j1} - p_{j1-1}) \cdot \ldots \cdot \sum_{jn}^{m} (p_{jn} - p_{jn-1}) \\
&= (p_{m1} - p_{01}) \cdot \ldots \cdot (p_{mn} - p_{0n}) \\
&= (b_1 - a_1) \cdot \ldots \cdot (b_n - b_1) \\
&= \prod_{i=1}^{n} (b_i - a_i) = v(I)
\end{aligned}
$$

$\Leftrightarrow m(I) = v(I)$