For any $f, g \in \mathcal{H}$ one has

$$|\langle f, g\rangle| \leq \|f\|\|g\| \qquad \text{Schwarz inequality,}$$

$$\|f+g\| \leq \|f\|+\|g\| \qquad \text{triangle inequality,}$$

$$\|f+g\|^2 \leq 2\|f\|^2+2\|g\|^2, \quad (1)$$

$$\big|\|f\|-\|g\|\big| \leq \|f-g\|. \quad (2)$$

$$\|f\| := \sqrt{\langle f, f\rangle}^{1/2}$$

In particular, note that for any $f, g, h \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ the following properties hold:

1. $\langle f, g\rangle = \overline{\langle g, f\rangle}$,
2. $\langle f, g+\lambda h\rangle = \langle f, g\rangle + \lambda\langle f, h\rangle$, but $\langle f+\lambda g, h\rangle = \langle f, h\rangle + \overline{\lambda}\langle g, h\rangle$,
3. $\|f\|^2 = \langle f, f\rangle \geq 0$, and $\|f\| = 0$ if and only if $f = 0$.

- Schwarz inequality

if $g = 0 \Leftrightarrow \|g\| = 0$

$|\langle f, 0\rangle| = 0 = \|f\| \cdot \|g\|$ ✓

if $g \neq 0$, set $\alpha = -\frac{\langle f, g\rangle}{\|g\|^2}$

$$0 \leq \|f+\alpha g\|^2 = \langle f+\alpha g, f+\alpha g\rangle$$

$$= \langle f, f\rangle + \langle f, \alpha g\rangle + \langle \alpha g, f\rangle + \langle \alpha g, \alpha g\rangle$$

$$= \|f\|^2 - \frac{\overline{\langle f, g\rangle}}{\|g\|^2}\langle f, g\rangle - \frac{\langle f, g\rangle}{\|g\|^2}\overline{\langle f, g\rangle} + \frac{\overline{\langle f, g\rangle}\langle f, g\rangle}{\|g\|^2}$$

$\cdot \|g\|^2$

$$\Leftrightarrow 0 \leq \|f\|^2\|g\|^2 - \overline{\langle f, g\rangle}\langle f, g\rangle$$

$$\Leftrightarrow |\langle f, g\rangle|^2 \leq \|f\|^2\|g\|^2$$

Since $|\langle f, g\rangle| \geq 0$ and $\|f\|, \|g\| \geq 0$, one can take the square root, so

$$|\langle f, g\rangle| \leq \|f\|\|g\|$$

→ if it's equality, so $|\langle f, g\rangle| = \|f\| \, \|g\|$, either f or $g = 0$ like shown above, or f and g are linear dependent, which means $f = a \cdot g$, with $a \in \mathbb{C}$

$$\Rightarrow \quad |\langle f, g\rangle| = |\langle ag, g\rangle| = |a| \langle g, g\rangle = |a| \, \|g\|^2$$
$$= \|f\| \, \|g\|$$

- triangle inequality

$$\|f+g\|^2 = \langle f+g, f+g\rangle$$
$$= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$$

take absolute of both sides

$$\|f+g\|^2 = \big|\, \|f\|^2 + \|g\|^2 + \langle f,g\rangle + \langle g,f\rangle \,\big|$$

use triangle eq. for skalars

$$\leq \|f\|^2 + \|g\|^2 + |\langle f,g\rangle| + |\langle g,f\rangle|$$

since $|\langle g,f\rangle| = |\overline{\langle g,f\rangle}| = |\langle f,g\rangle|$

$$= \|f\|^2 + \|g\|^2 + 2|\langle f,g\rangle|$$

use Schwarz ineq.: $|\langle f,g\rangle| \leq \|f\| \cdot \|g\|$

$$\leq \|f\|^2 + \|g\|^2 + 2\|f\|\|g\|$$
$$= (\|f\| + \|g\|)^2$$

$$\Leftrightarrow \|f+g\|^2 \leq (\|f\| + \|g\|)^2$$

Since both sides are positive, one can take the square root

$$\|f+g\| \leq \|f\| + \|g\|$$

• (1) $\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$

$$\|f+g\|^2 = \langle f+g, f+g \rangle$$
$$= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$$
$$= \|f\|^2 + \|g\|^2 + \langle f,g\rangle + \langle g,f\rangle$$

From $0 \leq \|f-g\|^2 = \|f\|^2 + \|g\|^2 - \langle f,g\rangle - \langle g,f\rangle$

we get $\langle g,f\rangle + \langle f,g\rangle \leq \|f\|^2 + \|g\|^2$.

So

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + \langle f,g\rangle + \langle g,f\rangle$$
$$\leq \|f\|^2 + \|g\|^2 + \|f\|^2 + \|g\|^2$$
$$= 2\|f\|^2 + 2\|g\|^2$$

• (2) $\big|\|f\| - \|g\|\big| \leq \|f-g\|$

suppose w.l.o.g. $\|f\| \geq \|g\|$

$$\big|\|f\| - \|g\|\big| = \|f\| - \|g\| = \|f-g+g\| - \|g\|$$
$$\overset{\Delta\text{-ineq.}}{\leq} \|f-g\| + \|g\| - \|g\|$$

$\Leftrightarrow \big|\|f\| - \|g\|\big| \leq \|f-g\|$