$$[\mathcal{F} f](\xi) \equiv \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X)\,dX$$, $f \in L^1(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$

0. $|[\mathcal{F}f](\xi)| < \infty$

$$|[\mathcal{F}f](\xi)| = \frac{1}{\sqrt{2\pi}^n} \left| \int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X)\,dX \right|$$

Triangle ineq. $|x+y| \leq |x|+|y|$

$$\leq \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} |e^{-i\xi\cdot X} f(X)|\,dX$$

$$= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}_n} |f(X)|\,dX < \infty$$

since $f \in L^1(\mathbb{R}^n)$ is $\int_{\mathbb{R}_n} |f(x)|\,dx < \infty$

1. $\mathcal{F}$ is a linear map on $L^1(\mathbb{R}^n)$. , $f, g \in L^1(\mathbb{R}^n)$, $\lambda \in \mathbb{R}^n$

$$[\mathcal{F}(f+\lambda g)](\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}_n} e^{-i\xi X} (f(X) + \lambda g(X))\,dX$$

$$= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}_n} e^{-i\xi X} f(X)\,dX + \lambda \cdot \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi X} g(X)\,dX$$

$$= [\mathcal{F}f](\xi) + \lambda \cdot [\mathcal{F}g](\xi)$$

2. $|\hat{f}(\xi)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(X)|\,dX,$

Triangle ineq.

$$|\hat{f}(\xi)| = \frac{1}{\sqrt{2\pi}^n} \left| \int_{\mathbb{R}^n} e^{-iX\xi} f(X)\,dX \right| \leq \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}_n} |e^{-iX\xi}|\,|f(X)|\,dX$$

$$= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} |f(X)|\,dX$$

3. $\hat{f}$ belongs to $C_0(\mathbb{R}^n)$, meaning that $\hat{f}$ is a continuous function on $\mathbb{R}^n$ satisfying $\lim_{\|\xi\|\to\infty} \hat{f}(\xi) = 0$,

since $f \in L^1$, it can be approximated by a compactly supported continuous function $g \in L^1$.

$\Rightarrow \|f-g\|_{L^1} \leq \varepsilon$, $\varepsilon > 0$

triangle ineq.: $\|f\|_{L^1} \leq \|f-g\|_{L^1} + \|g\|_{L^1}$

for $g \in L_1$ continuous

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi x} g(x)\, dx$$

subsitute $x \to x + \frac{\pi}{\xi}$ for $\xi \neq 0$

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi(x+\frac{\pi}{\xi})} g(x+\tfrac{\pi}{\xi})\, dx \qquad e^{-i\pi} = -1$$

$$= \frac{-1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi x} g(x+\tfrac{\pi}{\xi})\, dx$$

$$\hat{g}(\xi) = \frac{1}{2} \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi x} [g(x) - g(x+\tfrac{\pi}{\xi})]\, dx$$

$$\lim_{\|\xi\|\to\infty} \hat{g}(\xi) = \frac{1}{2} \frac{1}{\sqrt{2\pi}^n} \lim_{\|\xi\|\to\infty} \int_{\mathbb{R}^n} e^{-i\xi x} [g(x) - g(x+\tfrac{\pi}{\xi})]\, dx = 0 \quad *$$

since $\lim_{\|\xi\|\to\infty} (g(x) - g(x+\frac{\pi}{\xi})) \to 0 \;\; \forall x \in \mathbb{R}^n$

$$\lim_{\|\xi\|\to\infty} |\hat{f}(\xi)| \leq \frac{1}{\sqrt{2\pi}^n} \lim_{\|\xi\|\to\infty} \left\| \int_{\mathbb{R}^n} e^{-i\xi x} (f(x) - g(x))\, dx + \int_{\mathbb{R}^n} e^{-i\xi x} g(x)\, dx \right\|$$

triangle ineq.

$$\leq \frac{1}{\sqrt{2\pi}^n} \lim_{\|\xi\|\to\infty} \Big(\underbrace{\int_{\mathbb{R}^n} \underbrace{|e^{-i\xi x}|}_{1} |f(x) - g(x)|\, dx}_{\|f-g\|_{L^1} \leq \varepsilon} + \underbrace{\left| \int_{\mathbb{R}^n} e^{-i\xi x} g(x)\, dx \right|}_{* \to 0} \Big)$$

$\leq \varepsilon$

$\Rightarrow \lim_{\|\xi\|\to\infty} |\hat{f}(\xi)| \to 0$, since ε abitrary

Now to show that $\hat{f}(\xi)$ is continuous, we show hat $|\hat{f}(\xi)-\hat{f}(\xi+\delta)| \to 0$ for $\delta \to 0$:

$$|\hat{f}(\xi)-\hat{f}(\xi+\delta)| = \frac{1}{\sqrt{2\pi}^n} \left| \int_{\mathbb{R}^n} e^{-i\xi x} f(x)\,dx - \int_{\mathbb{R}^n} e^{-i(\xi+\delta)x} f(x)\,dx \right|$$

$$= \frac{1}{\sqrt{2\pi}^n} \left| \int_{\mathbb{R}^n} f(x) \left(e^{-i\xi x} - e^{-i(\xi+\delta)x} \right) dx \right|$$

triangle inequality

$$\leq \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} |f(x)|\, |e^{-i\xi x} - e^{-i(\xi+\delta)x}|\, dx$$

$$= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} |f(x)|\, \underbrace{|e^{-i\xi x}|}_{1}\, |1-e^{-i\delta x}|\, dx$$

$$= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} |f(x)|\, |1-e^{-i\delta x}|\, dx$$

$$= \frac{1}{\sqrt{2\pi}^n} \left(\int_{\mathbb{R}^n / B_r(0)} |f(x)|\, \underbrace{|1-e^{-i\delta x}|}_{\leq 2}\, dx + \int_{B_r(0)} |f(x)|\, |1-e^{-i\delta x}|\, dx \right)$$

$$\leq \frac{1}{\sqrt{2\pi}^n} \left(2 \cdot \int_{\mathbb{R}^n / B_r(0)} |f(x)|\, dx + \int_{B_r(0)} |f(x)|\, |1-e^{-i\delta x}|\, dx \right)$$

$$f \in L^1_{loc}(\mathbb{R}) \Rightarrow \int_{\mathbb{R}^n / B_r(0)} |f(x)|\, dx < \frac{\varepsilon}{2}$$

$$\leq \frac{1}{\sqrt{2\pi}^n} \left(\varepsilon + \int_{B_r(0)} |f(x)|\, \underbrace{|1-e^{-i\delta x}|}_{\xrightarrow[\delta \to 0]{} 0}\, dx \right)$$

$$= \frac{\varepsilon}{\sqrt{2\pi}^n} = \varepsilon'$$

$$|\hat{f}(\xi)-\hat{f}(\xi+\delta)| \leq \varepsilon'$$

$\Leftrightarrow$ $\hat{f}(\xi)$ is continuous

4. $\mathcal{F}(f * g) \equiv \widehat{f * g} = \hat{f}\,\hat{g}$, where the convolution of f and g is defined by

$$[f * g](X) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(X - Y)\, g(Y)\, \mathrm{d}Y,$$

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-iX\cdot\xi} \left(\frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(X-Y)\, g(Y)\, dY \right) dX$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iX\xi} f(X-Y)\, g(Y)\, dY\, dX$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-iX\xi} f(X-Y)\, dX \right) g(Y)\, dY$$

substitute $X - Y = Z$, $\frac{dZ}{dX} = 1$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-i(Z+Y)\xi} f(Z)\, dZ \right) g(Y)\, dY$$

$$= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{-iY\xi} g(Y)\, dY$$

$$= \hat{f}(\xi)\, \hat{g}(\xi)$$

$$\int g f' = g \cdot f - \int g' f$$

5. If $\partial_j f$ exists and belongs to $L^1(\mathbb{R}^n)$, then

$$[\mathcal{F}(-i\partial_j f)](\xi) \equiv [\widehat{-i\partial_j f}](\xi) = \xi_j \hat{f}(\xi).$$

$$[\mathcal{F}(-i\partial_j f)](\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-iX\xi} (-i\partial_j f(X))\, dX$$

$= 0$ since $f(X) \in L^1$

partial integration

$$\stackrel{\downarrow}{=} \frac{1}{\sqrt{2\pi}^n} \Bigg(\underbrace{\int \cdots \int}_{n-1 \text{ times}} -i e^{-i\Sigma_k X_k \xi_k} f(x_1 \ldots x_n) \Big|_{x_j = \pm\infty} dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n$$

$$+ i \int_{\mathbb{R}^n} (-i\xi_j) e^{-iX\xi} f(X)\, dX \Bigg)$$

$$= \xi_j \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-iX\xi} f(X)\, dX = \xi_j \hat{f}(\xi)$$