- Check $\|f\| := \sqrt{\langle f, f \rangle}$ is a norm

- $\|f\| \geq 0$ and $\|f\| = 0$ iff $f = 0$

$\|f\| = \sqrt{\langle f, f \rangle}$

since $\langle f, f \rangle \geq 0$

$\sqrt{\langle f, f \rangle} \geq 0$

$\therefore$

$\|f\| \geq 0$

$\Leftarrow$ assume $\|f\| = 0$

$0 = \sqrt{\langle f, f \rangle}$

$\langle f, f \rangle = 0$ by definition

$\Rightarrow$ Assume $f = 0$

$\therefore$

$\langle f, f \rangle = 0$ by definition it implies $f = 0$

$\|f\| = \sqrt{\langle f, f \rangle}$

$\|f\| = \sqrt{0}$

$\|f\| = 0$

$\therefore$

$\|f\| \geq 0$ and $\|f\| = 0$ iff $f = 0$

- $\|\lambda f\| = |\lambda| \, \|f\|$

$\|\lambda f\| = \sqrt{\langle \lambda f, \lambda f \rangle}$

$\|\lambda f\|^2 = \langle \lambda f, \lambda f \rangle$

$= \lambda \langle \lambda f, f \rangle$

$= \lambda \bar{\lambda} \langle f, f \rangle$

$\|\lambda f\|^2 = |\lambda|^2 \langle f, f \rangle$

$\|\lambda f\| = |\lambda| \sqrt{\langle f, f \rangle}$

$\|\lambda f\| = |\lambda| \, \|f\|$

- If complex it is a modulus
- If real it is an absolute value

Show

$\|f+g\| \leq \|f\| + \|g\| \quad \alpha \in \mathbb{C}$

and $f, g \in \mathcal{H}$

$\|f+\alpha g\|^2 = \langle f+\alpha g, f+\alpha g \rangle$

$\|f+\alpha g\|^2 = \|f\|^2 + \alpha \langle f,g \rangle + \bar{\alpha} \langle g,f \rangle + |\alpha|^2 \|g\|^2$

take $\alpha = 1$

$\|f+g\|^2 = \|f\|^2 + \langle f,g) + \langle g,f \rangle + \|g\|^2$

$\leq \|f\|^2 + |\langle f,g \rangle + |\langle g,f \rangle| + \|g\|^2$

$\leq \|f\|^2 + 2\|f\| \cdot \|g\| + \|g\|^2$ (by Schwarz inequality)

$\leq (\|f\| + \|g\|)^2$

$\|f+g\| \leq \|f\| + \|g\|$