3. *If $\mathcal{M}$ is a subset of $\mathcal{H}$, then*

$$\mathcal{M}^\perp := \{f \in \mathcal{H} \mid \langle f, g\rangle = 0, \forall g \in \mathcal{M}\}$$

is a closed subspace of $\mathcal{H}$.

Definition 3.1.8 (Subspace, closed subspace). *A subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is a linear subset of $\mathcal{H}$, or more precisely $\forall f, g \in \mathcal{M}$ and $\lambda \in \mathbb{C}$ one has $f + \lambda g \in \mathcal{M}$. The subspace $\mathcal{M}$ is closed if whenever a sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{M}$ converges to $f_\infty \in \mathcal{H}$, then $f_\infty \in \mathcal{M}$. In other words, a closed space contains its limit points.*

complete

Note that if $\mathcal{M}$ is closed, then $\mathcal{M}$ is a Hilbert space in itself, with the scalar product and norm inherited from $\mathcal{H}$. For the next examples, we recall that $f, g \in \mathcal{H}$ are said to be *orthogonal* if $\langle f, g\rangle = 0$. In this case, we write $f \perp g$.

Ex. 3.1.10

Linear : Let $f_1, f_2 \in M^\perp$ and $\lambda \in \mathbb{C}$

$$\langle f_1 + \lambda f_2, g\rangle = \langle f_1, g\rangle + \bar{\lambda}\langle f_2, g\rangle$$
$$= 0 + 0$$

$f_1 + \lambda f_2 \in M^\perp$

Thus, $M^\perp$ linear.

Consider a strongly converging sequence : $(f_j)_{j\in N} \subset M^\perp$

$s\text{-}\lim_{j\to\infty} f_j = f_\infty \in \mathcal{H}$ implies $w\text{-}\lim_{j\to\infty} f_j = f_\infty$

The scalar product is inherited from $\mathcal{H}$ so $\langle f, g\rangle = \overline{\langle g, f\rangle}$

$$\lim_{j\to\infty} \langle f_j - f_\infty, g\rangle = \lim_{j\to\infty} \overline{\langle g, f_j - f_\infty\rangle}$$
$$= \bar{0}$$
$$= 0$$

We obtain $\lim_{j\to\infty} \langle f_j, g\rangle - \langle f_\infty, g\rangle = 0$

$$\Rightarrow \lim_{j\to\infty} \langle f_j, g\rangle = \langle f_\infty, g\rangle$$

$\forall j \in \mathbb{N}$, $\langle f_j, g\rangle = 0$ so $\langle f_\infty, g\rangle = 0$

Thus $f_\infty \in M^\perp$

For any convergent sequence $(f_j)_{j\in N}$ in $M^\perp$, its limit point f_∞ is also in $M^\perp$.