Let us now start the construction of L^p-spaces, starting with $L^1(\Omega)$ for $\Omega \subset \mathbb{R}^n$ as mentioned above. Recall that if $f = g$ a.e., then $\int_\Omega f(X)\,\mathrm{d}X = \int_\Omega g(X)\,\mathrm{d}X$, see Corollary 2.4.5 when $\Omega = [a,b]$. For that reason, we would like to put such functions in an equivalence class: For any $f, g \in \mathcal{L}(\Omega)$, we write $f \sim g$ whenever $f = g$ a.e. The property of this relation is summarized in the following exercise.

do

Exercise 2.6.2. *Prove that the relation $\sim$ defines an* equivalence relation, *namely the following three properties are satisfied for any $f, g, h \in \mathcal{L}(\Omega)$:*

1) $f \sim f$ (reflexivity),
2) If $f \sim g$ then $g \sim f$ (symmetry),
3) If $f \sim g$ and $g \sim h$, then $f \sim h$ (transitivity).

Lemma 2.4.4. *Let $f \in \mathcal{L}^\infty([a,b])$ with $f = 0$ a.e. Then f is Lebesgue integrable, and one has $\int_a^b f(x)\,\mathrm{d}x = 0$.*

bc $m([a,b]) =$ 21 0 ?

Corollary 2.4.5. *Let $f, g \in \mathcal{L}^\infty([a,b])$ with $f = g$ a.e. and assume that f is Lebesgue integrable. Then g is also Lebesgue integrable, and $\int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x$.*

bc the integral of the difference = 0. Lemma 2.2.4

Let us state a related result, which is in fact used for the second part of the proof of Theorem 2.4.3.

Lemma 2.4.6. *Let $f \in \mathcal{L}^\infty([a,b])$ be Lebesgue integrable, and assume that $f \geq 0$ a.e. on $[a,b]$ and that $\int_a^b f(x)\,\mathrm{d}x = 0$. Then $f = 0$ a.e. on $[a,b]$.*

Definition 2.3.6 (Almost everywhere). *Consider $f, g : [a,b] \to \mathbb{R}$.*

a.e = almost everywhere

1. *We write $f = g$ a.e. if the set $\{x \in [a,b] \mid f(x) \neq g(x)\}$ has Lebesgue measure 0,*
2. *We write $f \leq g$ a.e. if the set $\{x \in [a,b] \mid f(x) > g(x)\}$ has Lebesgue measure 0.*

In both cases, we say that the relation holds almost everywhere.

Note that one can define similarly $f < g$ a.e., $f \geq g$ a.e., and $f > g$ a.e. The next statement reveals the importance of this concept, and already provides a glimpse about the generality we are dealing with.

Proposition 2.3.7. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue measurable function, and let $g = f$ a.e. Then g is also a Lebesgue measurable function on $[a,b]$.*

1) $f = f \Rightarrow f \sim f$

2) Suppose $f = g$ a.e. Let us define $h = f - g$.
Then $h = 0$ a.e so $\int_\Omega h(x)\,dx = 0$

$\Rightarrow \int_\Omega f(x) - g(x)\,dx = 0$

$\Rightarrow \int_\Omega f(x)\,dx - \int_\Omega g(x)\,dx = 0$

if $f \sim g$ then $g \sim f$

Given $f \sim g \Rightarrow \int_\Omega f(x)\,dx - \int_\Omega g(x)\,dx = 0$

$\Rightarrow \int g(x)\,dx - \int_\Omega f(x)\,dx = 0$

$\Rightarrow g \sim f$

3) if $\overset{①}{f \sim g}$, $\overset{②}{g \sim h}$ then $f \sim h$

Let us define $W_{fh} = \{ X \in \Omega \mid f(X) \neq h(X) \}$
and similarly, $W_{fg} = \{ X \in \Omega \mid f(X) \neq g(X) \}$,
$W_{gh} = \{ X \in \Omega \mid g(X) \neq h(X) \}$

For $X \in W_{fh}$, X must be also contained in either W_{fg} or W_{gh}
since there would be a contradiction

Suppose $x \in W_{fh}$, and $x \notin W_{fg} \cup W_{gh}$

since $x \notin W_{fg} \cup W_{gh}$,

$f(x) = g(x) = h(x) \quad \Rightarrow f(x) = h(x)$

which contradicts our previous statement that $f(x) \neq h(x)$.

Thus $W_{fh} \subset W_{fg} \cup W_{gh}$

$$\begin{aligned} m(W_{fh}) &\leq m(W_{fg} \cup W_{gh}) \\ &\leq m(W_{fg}) + m(W_{gh}) \\ &\leq 0 + 0 \end{aligned}$$

$$\begin{aligned} \int_\Omega f(x)\,dx - \int_\Omega h(x)\,dx &= \int_{W_{fh}} f(x) - h(x)\,dx \\ &= 0 \end{aligned}$$

$f \sim h$