Orthogonal projections

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1 Introduction

The aim of this report is to introduce a special class of bounded linear operators called orthogonal projection, and show a few properties and relations about this special operator.

2 Definition and basic properties

Let us start with the definition of orthogonal projection (based on Definition 3.3.4 of the lecture notes [1]).

Definition 2.1 (Projection) *An element* $P \in \mathcal{B}(\mathcal{H})$ *is an* orthogonal projection *if* $P = P^2 = P^*$.

Next, we will show that there is a one-to-one correspondence between the set of closed subspaces of $\mathcal{H}$ and the set of orthogonal projections in $\mathcal{B}(\mathcal{H})$. That is, for any orthogonal projection P one can define a closed subspace $M := P\mathcal{H}$, and for any closed subspace M one can find an orthogonal projection P such that $P\mathcal{H} = M$. To prove this, first let us define for any $f \in \mathcal{H}$ and P orthogonal projection

$$\begin{aligned} f_{\|} &:= P(f), \\ f_{\perp} &:= f - f_{\|}. \end{aligned}$$

Clearly, $f = f_{\|} + f_{\perp}$. Also, since $PP = P$,

$$P(f_{\perp}) = P(f - f_{\|}) = P(f) - P(f_{\|}) = P(f) - P(P(f)) = P(f) - P(f) = 0.$$

From this, one can define equivalence classes $[f]$ such that $f_{\|}$ is the same for every element in the same class. Because of this, the orthogonal projection of every element in the same equivalence class is the same. Also, in every equivalence class there is an element f_0 with $f_{0\perp} = 0$ (i.e. $P(f_0) = f_0$). Because of this, $M := P\mathcal{H}$ can also be defined as $M := \{f \in \mathcal{H} \mid P(f) = f\}$. Now consider the set $N = \{f \in \mathcal{H} | P(f) = 0\}$. Since $P^* = P$, for any $f \in N$, $g \in M$ we have

$$\langle f, g \rangle = \langle f, P(g) \rangle = \langle P^*(f), g \rangle = \langle P(f), g \rangle = \langle 0, g \rangle = 0$$

which means that $g \in N^{\perp}$ (the orthocomplement of N, $N^{\perp} := \{f \in \mathcal{H} | \langle f, g \rangle, \forall g \in N\}$). This means that $M \subset N^{\perp}$. Next, let $h \in N^{\perp}$ (meaning $\langle f, h \rangle = 0 \quad \forall f \in N$). Since $P(h_{\|}) = h_{\|}$ and $P(h_{\perp}) = 0$, $h_{\|} \in M$ and $h_{\perp} \in N$, which also means $\langle f, h_{\|} \rangle = 0$. From these

$$\begin{aligned} & 0 = \langle f, h \rangle = \langle f, h_{\|} + h_{\perp} \rangle = \langle f, h_{\|} \rangle + \langle f, h_{\perp} \rangle = \langle f, h_{\perp} \rangle \\ \iff & 0 = \langle f, h_{\perp} \rangle \\ \iff & h_{\perp} \in N^{\perp}. \end{aligned}$$

But the only element which can be both in N and its orthogonal complement $N^{\perp}$ is the 0 element, so $h_{\perp} = 0$. Therefore $h = h_{\|} + h_{\perp} = h_{\|} \Rightarrow h \in M$ for any $h \in N^{\perp}$. This means that $N^{\perp} \subset M$, therefore $M = N^{\perp}$. But, by Example 3.1.9 of the lecture note, the orthocomplement of any subset of $\mathcal{H}$ is a closed subspace, so M is a closed subspace.

Next, let M be a closed subspace with orthonormal basis (ONB) $B = (b_1, b_2, ...)$, that is a basis such that

$$\langle b_j, b_k \rangle = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

and define

$$\begin{aligned} P : \mathcal{H} &\to \mathcal{H} \\ f &\mapsto \sum_{j=1} \langle b_j, f \rangle \, b_j. \end{aligned}$$

Clearly, the image of this function is M. This map is linear since if $f, g \in \mathcal{H}$, $\lambda \in \mathbb{C}$, from the linearity of the scalar product

$$\begin{aligned} P(f + \lambda g) &= \sum_{j=1} \langle b_j, f + \lambda g \rangle \, b_j = \sum_{j=1} (\langle b_j, f \rangle + \lambda \langle b_j, g \rangle) \, b_j = \sum_{j=1} \langle b_j, f \rangle \, b_j + \lambda \langle b_j, g \rangle \, b_j \\ &= \sum_{j=1} \langle b_j, f \rangle \, b_j + \lambda \sum_{j=1} \langle b_j, g \rangle \, b_j = P(f) + \lambda P(g). \end{aligned}$$

We also have from the linearity of the scalar product

$$\begin{aligned} P(P(f)) &= \sum_{j=1} \left\langle b_j, \left(\sum_{k=1} \langle b_k, f \rangle \, b_k \right) \right\rangle b_j = \sum_{j=1} \left(\sum_{k=1} \langle b_j, \langle b_k, f \rangle \, b_k \rangle \right) b_j = \sum_{j=1} \left(\sum_{k=1} \langle b_k, f \rangle \langle b_j, b_k \rangle \right) b_j \\ &= \sum_{j=1} \langle b_j, f \rangle \, b_j = P(f) \end{aligned}$$

since $(b_1, b_2, ...)$ is orthonormal and the scalar product gives a number in $\mathbb{C}$. Also, from the properties of the scalar product

$$\begin{aligned} \langle f, P(g) \rangle &= \left\langle f, \sum_{j=1} \langle b_j, g \rangle \, b_j \right\rangle = \sum_{j=1} \langle f, \langle b_j, g \rangle \, b_j \rangle = \sum_{j=1} \langle f, b_j \rangle \langle b_j, g \rangle \,, \\ \langle P(f), g \rangle &= \left\langle \sum_{j=1} \langle b_j, f \rangle \, b_j, g \right\rangle = \sum_{j=1} \langle \langle b_j, f \rangle \, b_j, g \rangle = \sum_{j=1} \overline{\langle b_j, f \rangle} \langle b_j, g \rangle = \sum_{j=1} \langle f, b_j \rangle \langle b_j, g \rangle \,, \end{aligned}$$

which means that $P = P^*$. From these we can conclude that $P = PP = P^*$, so P is an orthogonal projection (corresponding to M).
Finally, one should note that if P and Q are orthogonal projections, then PQ and QP are not orthogonal projections in general. Take the following as a simple counterexample. Let $\mathcal{H} = \mathbb{R}^2$, P be the orthogonal projection onto the x axis ($y = 0$), and Q be the orthogonal projection onto the $y = x$ line. We have (also see Figure 1)

$$\begin{aligned} P(4,0) &= (4,0), \\ QP(4,0) &= Q(4,0) = (2,2), \\ PQP(4,0) &= P(2,2) = (2,0), \\ QPQP(4,0) &= Q(2,0) = (1,1) \neq (2,2) = QP(4,0), \end{aligned}$$

therefore $(QP)(QP) \neq QP \Rightarrow QP$ is not a projection.

3 Some relations of projections

In this section, let us see some relationships between orthogonal projections and the corresponding subspaces. The statements that will be proven later are summarized in the following proposition (Exercise 3.3.6 from the lecture notes).

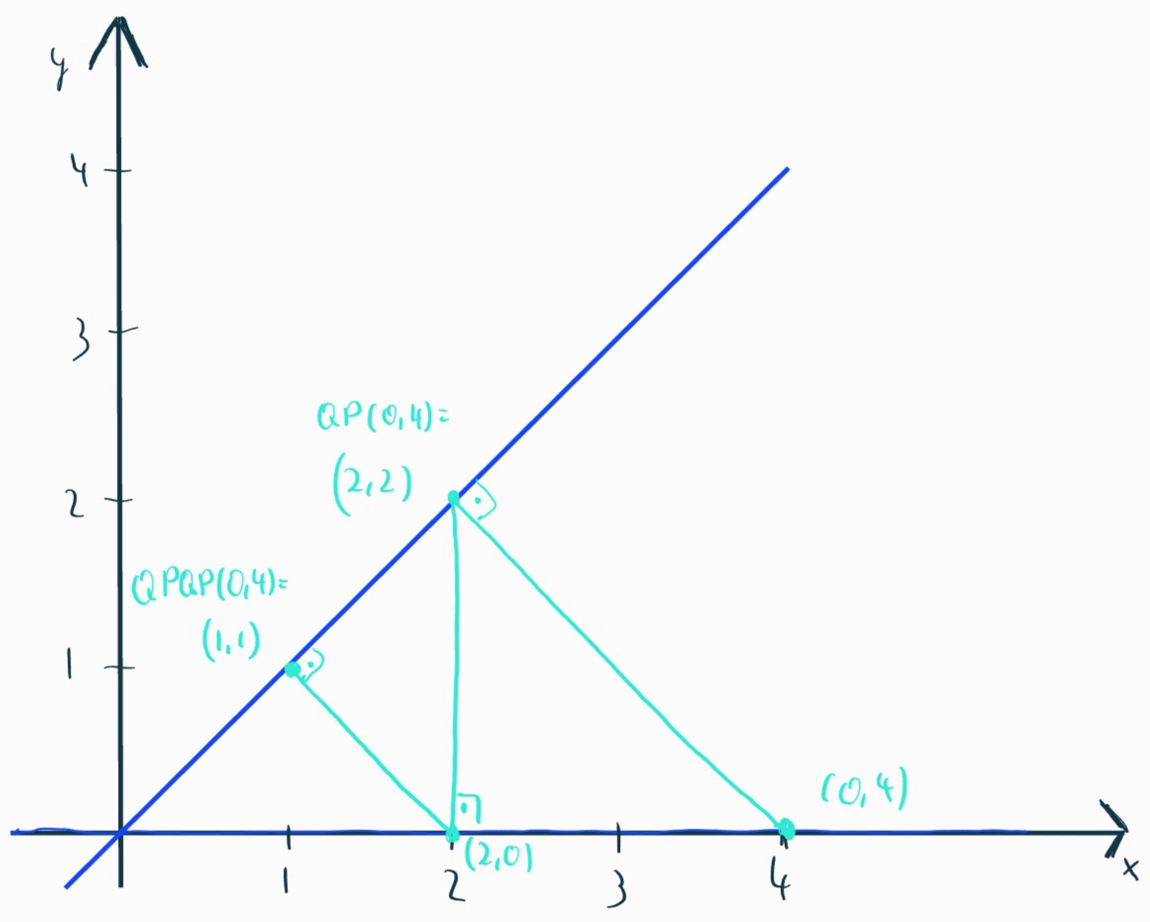

Figure 1: Counterexample for PQ or QP not being an orthogonal projection (where P, Q are orthogonal projections).

Proposition 3.1 *Let M and N be closed subspaces of $\mathcal{H}$ and P_M, P_N be the corresponding orthogonal projections.*

1. *If $P_M P_N = P_N P_M$, then $P_M P_N$ is a projection and the associated closed subspace is $M \cap N$.*
2. *If $M \subset N$, then $P_M P_N = P_N P_M = P_M$.*
3. *If $M \perp N$, then $P_M P_N = P_N P_M = \mathbf{0}$, and $P_{M \oplus N} = P_M + P_N$.*
4. *If $P_M P_N = \mathbf{0}$, then $M \perp N$.*

The proofs of these statements:

1. If $P_M P_N = P_N P_M$, then, since P_M, P_N are orthogonal projections

$$(P_M P_N)(P_M P_N) = P_M(P_N P_M)P_N = P_M(P_M P_N)P_N = (P_M P_M)(P_N P_N) = P_M P_N.$$

Also, from the properties of the adjoint

$$(P_M P_N)^* = P_N^* P_M^* = P_N P_M = P_M P_N.$$

Therefore, $P_M P_N = (P_M P_N)(P_M P_N) = (P_M P_N)^*$, so $P_M P_N$ is an orthogonal projection. Let $P := P_M P_N$. For any $f \in \mathcal{H}$, $P_M(f) \in M$ and $P_N(f) \in N$. From this, $P_M(P_N(f)) \in M$ and $P_N(P_M(f)) \in N$. But, since $P_M(P_N(f)) = P_N(P_M(f))$, $P(f) \in M$ and $P(f) \in N$, hence $P(f) \in M \cap N$. Also, since for any $g \in M \cap N$, $P(g) = g$, therefore the image of P (so the corresponding closed subspace) is $M \cap N$.

2. For any $f \in \mathcal{H}$ we have $P_M(f) \in M$ and $P_N(f) \in N$. Since $M \subset N$, $P_M(f) \in N$, so $P_N P_M(f) = P_M(f)$. Also, we can write $f = f_{\|} + f_{\perp} = f'_{\|} + f'_{\perp}$ with $P_M(f) = P_M(f_{\|}) = f_{\|}, P_M(f_{\perp}) = 0, P_N(f) = P_N(f'_{\|}) = f'_{\|}, P_N(f'_{\perp}) = 0$. Since $P_N(f'_{\perp}) = 0$, $f'_{\perp} \in \{f \in \mathcal{H} | P_N(f) = 0\}$, but we saw that any element of N is orthogonal to any element of this set, and any element of M is also an element of N, so any element of this set is orthogonal to M. Hence, $P_M(f'_{\perp}) = 0$. Using these and the linearity of the projection

$$\begin{aligned} P_M P_N(f) &= P_M P_N(f'_{\|} + f'_{\perp}) = P_M(P_N(f'_{\|}) + P_N(f'_{\perp})) = P_M(f'_{\|}) \\ &= P_M(f_{\|} + f_{\perp} - f'_{\perp}) = P_M(f_{\|}) + P_M(f_{\perp}) - P_M(f'_{\perp}) = f_{\|} = P_M(f) \end{aligned}$$

so $P_M P_N = P_N P_M = P_M$.

3. $M \perp N \iff \forall f \in M, g \in N,\ f \perp g \iff \forall f \in M, g \in N, P_M(g) = P_N(f) = 0, P_M(f) = f, P_N(g) = g$. Also, for any $h \in \mathcal{H}$, $P_M(h) \in M$ and $P_N(h) \in N$. Hence,

$$\begin{aligned} P_M P_N(h) &= \mathbf{0}, \\ P_N P_M(h) &= \mathbf{0}. \end{aligned}$$

Let $B_1 = (b_1, b_2, ...)$ ONB of N and $B_2 = (c_1, c_2, ...)$ ONB of M. Let $f \in M$ and also $f \in N$. We have $P_M(f) = f$ since $f \in M$, and $P_M(f) = 0$, since $f \in N$. Hence, $M \cap N = \{0\}$. Also, since $\forall f \in M, g \in N,\ f \perp g$, so $b_1, b_2, ...$ are all orthogonal to any of c_1, c_2. Also, since all these elements have norm 1, these elements are orthonormal, so $(b_1, b_2, ...) \cup (c_1, c_2, ...)$ is an ONB of $M \oplus N$. From how the orthogonal projection was defined

$$\begin{aligned} P_N(f) &= \sum_{j=1} \langle b_j, f \rangle b_j, \\ P_M(f) &= \sum_{j=1} \langle c_j, f \rangle c_j, \end{aligned}$$

and also

$$\begin{aligned} P_{M \oplus N} &= \sum_{j=1,k=1} \langle b_j, f \rangle b_j + \langle c_k, f \rangle c_k = \langle b_1, f \rangle b_1 + \langle b_2, f \rangle b_2 + ... + \langle c_1, f \rangle c_1 + \langle c_2, f \rangle c_2 + ... \\ &= \sum_{j=1} \langle b_j, f \rangle b_j + \sum_{j=1} \langle c_j, f \rangle c_j = P_N(f) + P_M(f). \end{aligned}$$

4. Let $f \in N$.

$$\begin{aligned} &P_M P_N(f) = \mathbf{0} \\ &\Rightarrow P_M(f) = \mathbf{0} \end{aligned}$$

so $\forall f \in N, P_M(f) = \mathbf{0} \Rightarrow M \perp N$.

4 Summary

In this report first orthogonal projections were defined, and then some basic properties of these operators were presented. One of these properties was that the product of two orthogonal projections is not an orthogonal projection in general. However, in the second half of the report we saw that although in general the product of two orthogonal projections is not an orthogonal projection, some relations still can be drawn between two orthogonal projections (and their corresponding closed subspaces).

5 References

1. Functional analysis lecture notes