Eigenvalues of multiplication operators

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July 28, 2023

1 Introduction

In this report first multiplication operators and the eigenvalues of a closed linear operator will be defined, then the condition for the multiplication operator to have eigenvalue(s) will be analyzed. Finally, examples of multiplication operators having (at least one) eigenvalue(s) will be presented.

2 Definitions

Let us start with the definition of the multiplication operator and the eigenvalues of a closed linear operator (based on Exercise 3.2.6 and Definition 3.6.5 of the lecture notes [1]).

Definition 2.1 *Let $\mathcal{H}$ be a Hilbert space and $\varphi \in L^\infty(\mathbb{R}^n)$. Let us set M_φ as $[M_\varphi f](X) := \varphi(X)f(X)$ for all $f \in \mathcal{H}$ and a.e. $X \in \mathbb{R}^n$.*

Definition 2.2 *For a closed linear operator A, a value $z \in \mathbb{C}$ is an* eigenvalue *of A if there exists $f \in \mathcal{D}(A), f \neq 0$ such that $Af = zf$. In such case, the element f is called an* eigenfunction *or* eigenvactor *of A associated with the eigenvalue z. The set of all eigenvalues of A is denoted by $\sigma_p(A)$.*

Note that $f \neq 0$ means that f is not 0 on a set with Lebesgue measure bigger than 0.

3 Condition for the multiplication operator to have eigenvalues

Let us look at the condition for a number z to be an eigenvalue. From the definition of eigenvalue

$$\begin{aligned}
& M_\varphi f = zf & \\
\iff\ & [M_\varphi f](X) = zf(X) & \text{a.e.} X \in \mathbb{R}^n \\
\iff\ & \varphi(X)f(X) = zf(X) & \text{a.e.} X \in \mathbb{R}^n \\
\iff\ & (\varphi(X) - z)f(X) = 0 & \text{a.e.} X \in \mathbb{R}^n \\
\Rightarrow\ & (\varphi(X) - z) = 0 & \text{a.e.} X \in \mathbb{R}^n, f(X) \neq 0 \\
\iff\ & \varphi(X) = z & \text{a.e.} X \in \mathbb{R}^n, f(X) \neq 0.
\end{aligned}$$

From this one can conclude that M_φ has an eigenvalue if $\exists \Omega \in \mathbb{R}^n$ with $m(\Omega) > 0$ such that φ is constant on Ω. The eigenvalue is going to be this constant and the corresponding eigenfunction is a function which is not 0 only on Ω or a subset of Ω.

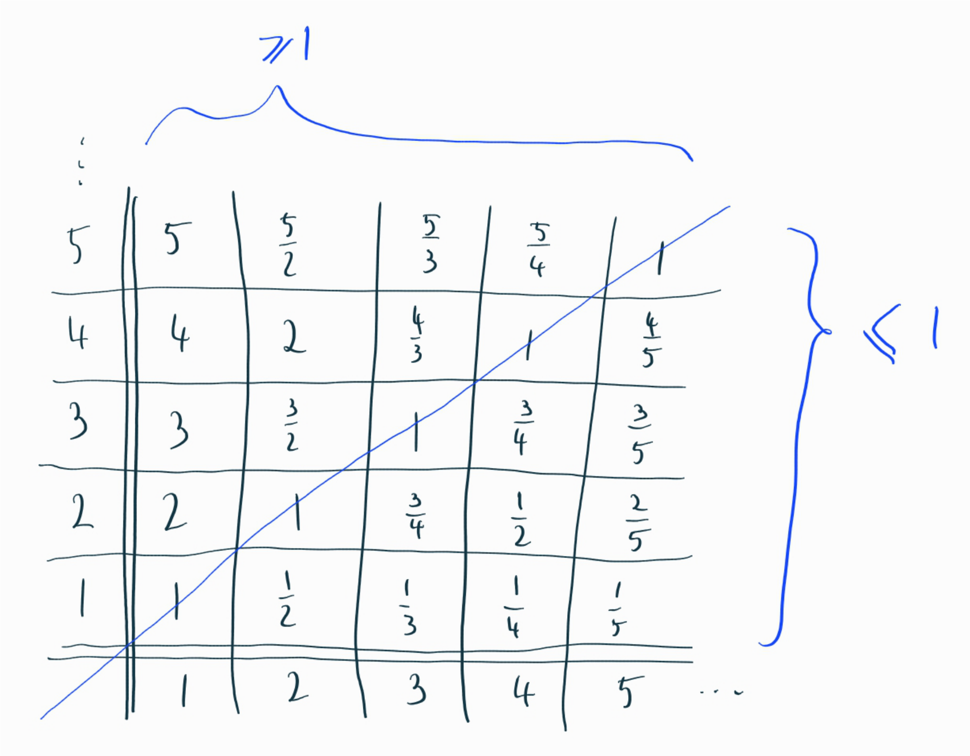
4 Examples of multiplication operators having eigenvalues

1. Let $\phi : \mathbb{R} \ni x \mapsto 1 \in \mathbb{R}$. We have

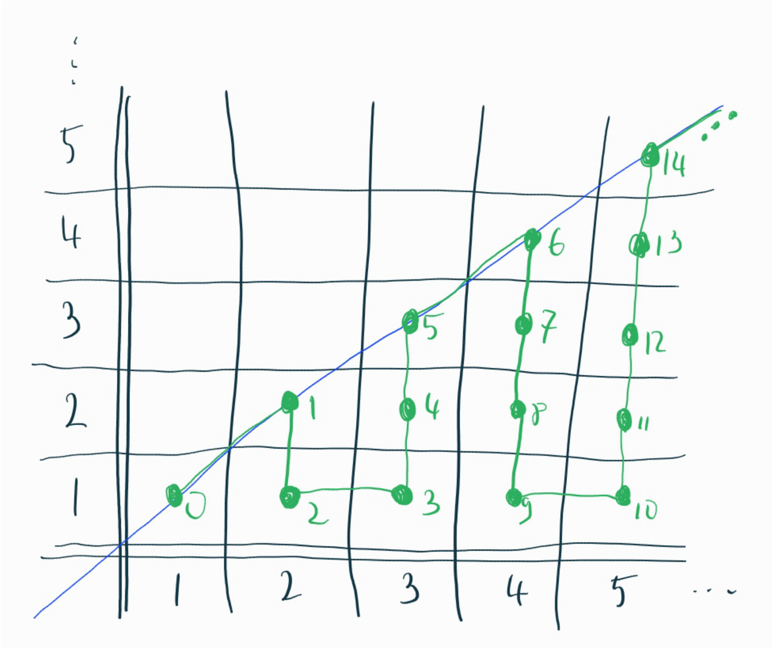
$$\begin{aligned}
M_\varphi f &= zf \\
\iff [M_\varphi f](X) &= zf(X) && \text{a.e.} X \in \mathbb{R}^n \\
\iff \varphi(X)f(X) &= zf(X) && \text{a.e.} X \in \mathbb{R}^n \\
\iff f(X) &= zf(X) && \text{a.e.} X \in \mathbb{R}^n \\
\iff (1-z)f(X) &= 0 && \text{a.e.} X \in \mathbb{R}^n.
\end{aligned}$$

   Since $f \neq 0$ a.e., we have that $z = 1$ is an eigenvalue (and the only eigenvalue).

2. Consider the following table:



   In every cell of the table we have the ratio of the number of the row and the number of the column. One can observe that the part of table below the line contains all rational numbers in the set $[0, 1]$. Now consider the following path:



and consider the following function

$$\begin{aligned} \psi : \mathbb{N} &\to [0,1] \cap \mathbb{Q} \\ n &\mapsto n\text{th number on the path.} \end{aligned}$$

One can observe that this is a surjective function from $\mathbb{N}$ to $[0,1] \cap \mathbb{Q}$. Then, consider

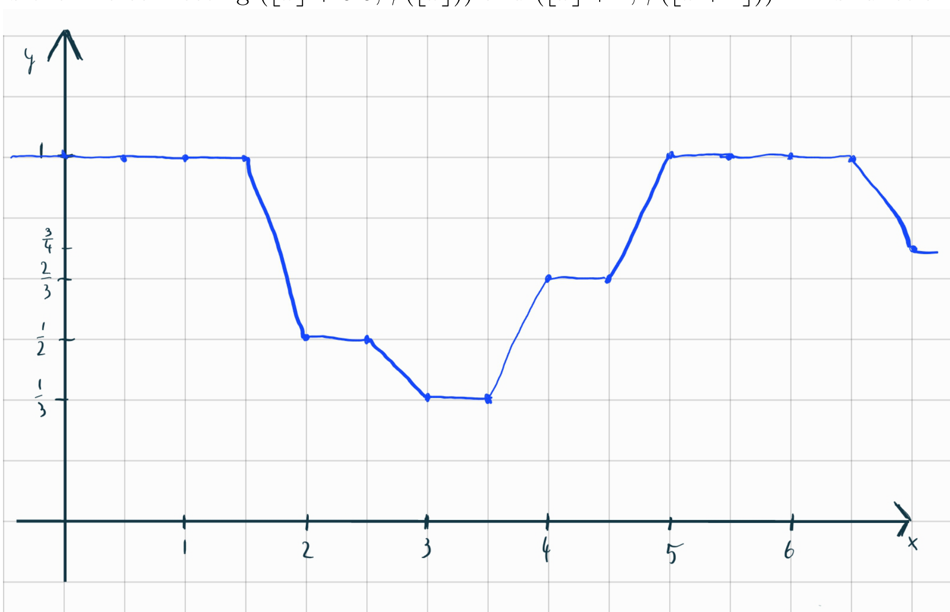
$\varphi : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \varphi(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \psi(\lfloor x \rfloor) & \text{if } x \in (\lfloor x \rfloor, \lfloor x \rfloor + 0.5] \\ \psi(\lfloor x \rfloor) + 2(\psi(\lfloor x+1 \rfloor) - \psi(\lfloor x \rfloor))(x - (\lfloor x \rfloor + 0.5)) & \text{if } x \in (\lfloor x \rfloor + 0.5, \lfloor x \rfloor + 1] \end{cases}$$

where $\lfloor x \rfloor$ is the floor function and

$$\psi(\lfloor x \rfloor) + \frac{\psi(\lfloor x+1 \rfloor) - \psi(\lfloor x \rfloor)}{0.5}(x - \lfloor x+0.5 \rfloor) = \psi(\lfloor x \rfloor) + 2(\psi(\lfloor x+1 \rfloor) - \psi(\lfloor x \rfloor))(x - (\lfloor x \rfloor + 0.5))$$

is the line connecting $(\lfloor x \rfloor + 0.5, \psi(\lfloor x \rfloor))$ and $(\lfloor x \rfloor + 1, \psi(\lfloor x+1 \rfloor))$. This function looks like



Clearly, the set of all eigenvalues of the corresponding multiplication operator is $\sigma_p = [0,1] \cap \mathbb{Q}$. An example of an eigenfunction corresponding to any $z \in \sigma_p$ is the characteristic function which is not 0 on the set $(n, n+0.5]$ with $\psi(n) = z$ (we can always find such n, since ψ is surjective).

5 Summary

First, multiplication operators and eigenvalues were defined, and then the condition for the multiplication operator to have eigenvalue(s) was analyzed, and finally two examples of multiplication operators with $\sigma_p \neq \emptyset$ were given.

6 References

1. Functional analysis lecture notes