L^p spaces with p smaller than 1

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1 Introduction

In the course L^p spaces were defined for $p \geq 1$, but one might wonder what happens when $0 < p < 1$. In this report first the definition of L^p spaces will be recalled, then it will be discussed what happens when we have $0 < p < 1$.

2 L^p spaces for $p \geq 1$

Firstly, let us recall the definition of L^p spaces for $p \geq 1$ (Definition 2.6.7 of [3]).

Definition 2.1 **(L^p spaces)** *For any $p \geq 1$ the set $L^p(\Omega)$ is defined as*

$$\{f : \Omega \to \mathbb{K} | f \text{ is Lebesgue measurable, and } |f|^p \in \mathcal{L}(\Omega)\} / \sim$$

endowed with the norm

$$||f||_p := \left(\int_\Omega |f(X)|^p \mathrm{d}X \right)^{\frac{1}{p}}.$$

Note that in L^p spaces we consider equivalence classes of functions instead of single functions (two functions are in the same equivalence group if they are equal a.e.). The norm must satisfy the following three properties by definition [3]

1. $||\lambda f||_p = |\lambda| ||f||_p$ for any $\lambda \in \mathbb{C}$ and $f \in L^p$,
2. $||f_1 + f_2||_p \leq ||f_1||_p + ||f_2||_p$ for any $f_1, f_2 \in L^p$,
3. $||f||_p = 0$ if and only if $f = 0$ a.e..

3 L^p spaces when $0 < p < 1$

The main reason the L^p space is not defined for these values of p is that the 2nd property of the norm from the previous section (the triangle inequality) is not always true. An example to this is provided in the followings (this example is based on the sketch provided in [1]). Take two bounded, disjoint Lebesgue-measurable sets $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^n$. Let a be the Lebesgue measure of Ω_1 and b be the Lebesgue measure of Ω_2. Next, consider the characteristic functions corresponding to these two sets:

$$f_1 : \mathbb{R}^n \to \mathbb{R} \qquad\qquad f_2 : \mathbb{R}^n \to \mathbb{R}$$

$$f_1(X) = \begin{cases} 1 & \text{if } X \in \Omega_1 \\ 0 & \text{if } X \notin \Omega_1 \end{cases} \qquad\qquad f_2(X) = \begin{cases} 1 & \text{if } X \in \Omega_2 \\ 0 & \text{if } X \notin \Omega_2 \end{cases}$$

These functions are in L^p (with $0 < p < 1$), since

$$\int_{\mathbb{R}^n} |f_1(X)|^p \mathrm{d}X = \int_{\Omega_1} 1^p \mathrm{d}X = \int_{\Omega_1} \mathrm{d}X = m(\Omega_1) = a < \infty,$$
$$\int_{\mathbb{R}^n} |f_2(X)|^p \mathrm{d}X = \int_{\Omega_2} 1^p \mathrm{d}X = \int_{\Omega_2} \mathrm{d}X = m(\Omega_2) = b < \infty,$$

since Ω_1 and Ω_2 are bounded. Therefore, the "norm" of these functions

$$||f_1||_p = \left(\int_{\mathbb{R}^n} |f_1(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} = a^{\frac{1}{p}},$$
$$||f_2||_p = \left(\int_{\mathbb{R}^n} |f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} = b^{\frac{1}{p}}.$$

The left side of the inequality in the properties of the norm (the norm of the sum the two functions)

$$||f_1 + f_2||_p = \left(\int_{\mathbb{R}^n} |f_1(X) + f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}}.$$

Since the sets are disjoint, $f_1(X) = 1$ and $f_2(X) = 0$ when $X \in \Omega_1$, $f_1(X) = 0$ and $f_2(X) = 1$ when $X \in \Omega_1$, and $f_1(X) = f_2(X) = 0$ when X is in neither of the two sets. Therefore

$$\begin{aligned}||f_1 + f_2||_p &= \left(\int_{\Omega_1} |1+0|^p \mathrm{d}X + \int_{\Omega_2} |0+1|^p \mathrm{d}X\right)^{\frac{1}{p}} = \left(\int_{\Omega_1} \mathrm{d}X + \int_{\Omega_2} \mathrm{d}X\right)^{\frac{1}{p}} \\ &= (m(\Omega_1) + m(\Omega_2))^{\frac{1}{p}} = (a+b)^{\frac{1}{p}}.\end{aligned}$$

Next, let us prove that $(a+b)^{\frac{1}{p}} \geq a^{\frac{1}{p}} + b^{\frac{1}{p}}$. For this let us define $f(t) := (1+t)^{\frac{1}{p}} - 1 - t^{\frac{1}{p}}$ (based on the proof of Lemma 1 in [5]). Then

$$f'(t) = \frac{(1+t)^{\frac{1}{p}-1}}{p} - \frac{t^{\frac{1}{p}-1}}{p} = \frac{1}{p}\left((1+t)^{\frac{1}{p}-1} - t^{\frac{1}{p}-1}\right) > 0$$

for all $t \in (0, \infty)$. This is because $(1+t) > t$, and $0 < p < 1 \Rightarrow \frac{1}{p} > 1 \Rightarrow \frac{1}{p} - 1 > 0$, so the function $g(x) = x^{\frac{1}{p}-1}$ is increasing. Therefore, $f(t)$ is increasing on $t \in (0, \infty)$. Also, $f(0) = 0$, so $f(t) \geq 0$ on $(0, \infty)$. Now let us substitute $t = \frac{a}{b}$. Note : $a, b \geq 0$, since the Lebesgue measure is always positive, and we can assume $b \neq 0$, since if $b = 0$ then it is trivial that the inequality is true (we get $a^{\frac{1}{p}} \geq a^{\frac{1}{p}}$).

$$\begin{aligned}&\left(1 + \frac{a}{b}\right)^{\frac{1}{p}} - 1 - \left(\frac{a}{b}\right)^{\frac{1}{p}} \geq 0 \\ \iff &\left(\frac{a+b}{b}\right)^{\frac{1}{p}} - 1 - \left(\frac{a}{b}\right)^{\frac{1}{p}} \geq 0 \\ \iff &(a+b)^{\frac{1}{p}} - (a^{\frac{1}{p}} + b^{\frac{1}{p}}) \geq 0 \\ \iff &(a+b)^{\frac{1}{p}} \geq a^{\frac{1}{p}} + b^{\frac{1}{p}}\end{aligned}$$

substituting back one gets

$$||f_1 + f_2||_p \geq ||f_1||_p + ||f_2||_p,$$

therefore the triangle inequality does not hold for these two functions.
Here, we saw that the p-norm does not define a norm; however, it does define a quasi-norm. Before proving this, let us see the definition of a quasi-norm [4].

Definition 3.1 *If V is a complex vector space, then a map $V \ni v \mapsto ||v|| \in [0, \infty)$ is a quasi-norm if:*

1. $||\lambda v|| = |\lambda| ||v||$ *for any* $\lambda \in \mathbb{C}$ *and* $v \in V$,
2. *there exists a real number C independent of v_1 and v_2 such that* $||v_1 + v_2|| \leq C(||v_1|| + ||v_2||)$ *for any* $v_1, v_2 \in V$,
3. $||v|| = 0$ *if and only if* $v = 0$.

First let us check if the first condition holds. Let $f \in L^p$ and $\lambda \in \mathbb{C}$. We have

$$||\lambda f||_p = \left(\int_\Omega |\lambda f(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} = \left(\int_\Omega |\lambda|^p |f(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} = |\lambda| \left(\int_\Omega |f(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} = |\lambda| ||f||_p$$

because of the linearity of the integral.
Next, let us check the third condition. Note that since in L^p spaces we consider equivalence classes of functions instead of single functions (and two functions are in the same equivalence group if they are equal a.e.), the condition modifies to: *for a function $f \in L^p$, $||f||_p = 0$ if and only $f = 0$ a.e.*. Since it is an equivalence, we have to check both directions

- If $f(X) = 0$ for a.e. X, then $|f(X)|^p$ is also 0 for a.e. X, so its Lebesgue integral is 0. From this,

$$||f||_p = \left(\int_\Omega |f(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} = 0^{\frac{1}{p}} = 0.$$

- $||f||_p = 0$. Let us use proof by contradiction, and let us assume that f is not 0 a.e.. By definition

$$||f||_p = \left(\int_\Omega |f(X)|^p \mathrm{d}X\right)^{\frac{1}{p}}.$$

  One can notice that because of the absolute value, $|f(X)|^p$ is never negative, and it is also positive on a set with Lebesgue measure bigger than 0. This means that the integral is bigger than 0, so the norm is bigger than 0. But, this contradicts the initial assumption, therefore $f(X)$ is 0 almost everywhere.

Finally, let us check the second condition, for which we follow [2]. First we show that $(a+b)^p \leq a^p + b^p$ for $0 < p < 1$ (using the same method that we used at the beginning of this section). Let us define $f(t) := (1+t)^p - 1 - t^p$ (also based on the proof of Lemma 1 in [5]). Then

$$f'(t) = p(1+t)^{p-1} - pt^{p-1} = p\left((1+t)^{p-1} - t^{p-1}\right) < 0$$

for all $t \in (0, \infty)$. This is because $(1+t) > t$, and $0 < p < 1 \Rightarrow p - 1 < 0$, so the function $g(x) = x^{p-1}$ is decreasing. Therefore, $f(t)$ is decreasing on $t \in (0, \infty)$. Also, $f(0) = 0$, so $f(t) \leq 0$ on $(0, \infty)$. Now let us substitute $t = \frac{a}{b}$. Note : $a, b \geq 0$, since the Lebesgue measure is always positive, and we can assume $b \neq 0$, since if $b = 0$ then it is trivial that the inequality is true (we get $a^p \leq a^p$).

$$\begin{aligned} &\left(1 + \frac{a}{b}\right)^p - 1 - \left(\frac{a}{b}\right)^p \leq 0 \\ \iff &\left(\frac{a+b}{b}\right)^p - 1 - \left(\frac{a}{b}\right)^p \leq 0 \\ \iff &(a+b)^p - (a^p + b^p) \leq 0 \\ \iff &(a+b)^p \leq a^p + b^p. \end{aligned}$$

Next, consider two functions f_1, f_2. We have

$$|f(X) + g(X)| \leq |f(X)| + |g(X)|.$$

Since taking the pth power (p positive) is increasing, and using the previous result

$$|f(X) + g(X)|^p \leq (|f(X)| + |g(X)|)^p \leq |f(X)|^p + |g(X)|^p,$$

from which

$$\int_\Omega |f_1(X) + f_2(X)|^p \mathrm{d}X \leq \int_\Omega |f_1(X)|^p \mathrm{d}X + \int_\Omega |f_2(X)|^p \mathrm{d}X. \tag{1}$$

Now, let for $c > 1$

$$f(t) = \frac{t^c + 1}{(t+1)^c}.$$

Let us calculate the derivative of this function and find where it is 0. The derivative

$$\begin{aligned} f'(t) &= \frac{ct^{c-1}(t+1)^c - (t^c+1)\cdot c(t+1)^{c-1}}{(t+1)^{2c}} = \frac{c}{(t+1)^{c+1}}(t^{c-1}(t+1) - (t^c+1)) \\ &= \frac{c}{(t+1)^{c+1}}(t^c + t^{c-1} - t^c - 1) = \frac{c}{(t+1)^{c+1}}(t^{c-1} - 1), \end{aligned}$$

and when $f'(t) = 0$

$$\begin{aligned} 0 &= \frac{c}{(t+1)^{c+1}}(t^{c-1} - 1) \\ \iff 0 &= t^{c-1} - 1 \\ \iff t^{c-1} &= 1 \\ \iff t &= 1. \end{aligned}$$

Let us calculate the second derivative at $t = 1$ to check if this is a minimum, maximum, or something else. The second derivative:

$$f''(t) = c\frac{(c-1)t^{c-2}(t+1)^{c+1} - (t^{c-1}-1)(c+1)(t+1)^c}{(t+1)^{2c+2}}.$$

The second derivative at $t = 1$:

$$f''(1) = c\frac{(c-1)1^{c-2}(1+1)^{c+1} - (1^{c-1}-1)(c+1)(1+1)^c}{(1+1)^{2c+2}} = c\frac{(c-1)2^{c+1}}{2^{2c+2}}$$

which is positive ($c > 1$), so this is a minimum. Since we only have this extremum, this minimum is unique. Therefore

$$\begin{aligned} & f(t) \geq f(1) \\ \iff & \frac{t^c+1}{(t+1)^c} \geq \frac{1^c+1}{(1+1)^c} = \frac{2}{2^c} = 2^{1-c} \\ & \Rightarrow t^c + 1 \geq 2^{1-c}(t+1)^c \end{aligned}$$

let $t = a/b$ with $a \geq 0$ and $b > 0$,

$$\begin{aligned} & \left(\frac{a}{b}\right)^c + 1 \geq 2^{1-c}\left(\frac{a}{b}+1\right)^c \\ \iff & a^c + b^c \geq 2^{1-c}(a+b)^c. \end{aligned}$$

Note that this inequality is also true when $b = 0$: $a^c \geq 2^{1-c}a^c \iff 1 = 2^0 \geq 2^{1-c} \iff 0 \geq 1 - c$, which is true since $c > 1$ (when $a = 0$, we get $0 \geq 0$, which is also true). Next, let $c = 1/p$ (for $0 < p < 1$), $a = \int_\Omega |f_1(X)|^p \mathrm{d}X$, $b = \int_\Omega |f_2(X)|^p \mathrm{d}X$. From the definition of the "norm"

$$\begin{aligned} & \left(\int_\Omega |f_1(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} + \left(\int_\Omega |f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} \geq 2^{1-\frac{1}{p}}\left(\int_\Omega |f_1(X)|^p \mathrm{d}X + \int_\Omega |f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} \\ \iff & \left(\int_\Omega |f_1(X)|^p \mathrm{d}X + \int_\Omega |f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1}\left(\left(\int_\Omega |f_1(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} + \left(\int_\Omega |f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}}\right). \end{aligned}$$

Based on (1) and that the function $x \mapsto x^{\frac{1}{p}}$ is increasing for positive p, we have

$$\left(\int_\Omega |f_1(X) + f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} \leq \left(\int_\Omega |f_1(X)|^p \mathrm{d}X + \int_\Omega |f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}}$$

which implies that

$$\left(\int_\Omega |f_1(X)+f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1}\left(\left(\int_\Omega |f_1(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} + \left(\int_\Omega |f_2(X)|^p \mathrm{d}X\right)^{\frac{1}{p}}\right)$$

$$\iff ||f_1+f_2||_p \leq 2^{\frac{1}{p}-1}\left(||f_1||_p + ||f_2||_p\right).$$

Another interesting property of the quasi-norm of L^p is the following inequality (called the reverse Minkowski's inequality, for the proof refer to [5])

Proposition 3.1 *(Reverse Milkowski's) Let $f_1, ..., f_n \in L^p$ where $0 < p < 1$. Then*

$$\sum_{j=1}^{n} ||f_j||_p \leq \left|\left|\sum_{j=1}^{n} |f_j|\right|\right|_p .$$

Note: setting $n = 2$ we can get

$$||f_1||_p + ||f_2||_p \leq |||f_1| + |f_2|||_p \leq 2^{\frac{1}{p}-1}\left(||f_1||_p + ||f_2||_p\right).$$

4 Conclusion

In this report L^p spaces were introduced for $p \geq 1$, and then it was investigated why it cannot be defined the same way for $0 < p < 1$. It turned out that the main reason L^p cannot be defined for $0 < p < 1$ is that the corresponding norm does not satisfy the triangle inequality, so it is not actually a norm. However, it also turned out that even though it is not a norm, it does satisfy the conditions of being a quasi-norm.

5 References

[1] Himanshu (https://math.stackexchange.com/users/519207/himanshu). *Show that L^p space for $0 < p < 1$ does not define a norm.* Mathematics Stack Exchange . URL:https://math.stackexchange.com/q/4360482 (version: 2022-01-19). eprint: `https://math.stackexchange.com/q/4360482`. URL: `https://math.stackexchange.com/q/4360482`.

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[3] *Introduction to functional analysis lecture notes.*

[4] N. Kalton. *Plurisubharmonic functions on quasi-Banach spaces.* Vol. 84. Studia Mathematica 3. Institute of Mathematics, Polish Academy of Sciences, 1986, pp. 297–324. DOI: `10.4064/sm-84-3-297-324`.

[5] Matt Rosenzweig. *L^p Spaces for $0 < p < 1$.* URL: `https://matthewhr.files.wordpress.com/2012/09/lp-spaces-for-p-in-01.pdf`.