On norm of $L^1(\Omega)$

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This report aims to give a proof for **Exercise 2.6.4**. We start by recalling all the discussed concepts.

L^1 – space

The set $L^1(\Omega)$ is defined by $\mathcal{L}(\Omega)/\sim$, namely the elements of $L^1(\Omega)$ consists of equivalence classes of Lebesgue integrable functions which are equal almost everywhere. The set $L^1(\Omega)$ is endowed with the norm:

$$\|f\|_1 \equiv \|f\|_{L^1(\Omega)} = \int_\Omega |f(X)| dX$$

Note that, since $L^1 := \mathcal{L}(\Omega)/\sim$, it consists of equivalence classes of functions that are equal almost everywhere. For example, f and g belong to the same equivalency class (or are treated as the same) if the set $\{x \in \Omega \mid f(x) \neq g(x)\}$ *has Lesbegue Measure 0*, then one writes $f(x) = g(x)$ a.e.

Exercise 2.6.4

Show that the map defined above $f \mapsto \|f\|_1$ defines a norm on $L^1(\Omega)$.

Proof. To show that the map $f \mapsto \|f\|_1$ defines a norm on $L^1(\Omega)$, we need to show that the map satisfies the three properties of a norm. Namely, if $p : X \to \mathbb{R}$ defines a norm on X then p satisfies the following:

1. **Subadditivity/Triangle inequality:** $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.
2. **Absolute homogeneity:** $p(sx) = |s|p(x)$ for all $x \in X$ and all $s \in \mathbb{C}$.
3. **Positive definiteness/positiveness:** for all $x \in X$, if $p(x) = 0$, then $x = 0$.

Subadditivity/Triangle inequality:
For any $f, g \in L^1(\Omega)$, we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \quad \text{for all x} \in \Omega.$$

Therefore, since the integral is linear, we have

$$\|f+g\|_1 = \int_\Omega |f(x) + g(x)|\, dx \leq \int_\Omega |f(x)|\, dx + \int_\Omega |g(x)|\, dx = \|f\|_1 + \|g\|_1.$$

Therefore, the subadditivity property holds.

Absolute homogeneity:
For any $f \in L^1(\Omega)$ and s with $s \in \mathbb{C}$, we have:

$$\|sf\|_1 = \int_\Omega |sf(x)|\, dx = \int_\Omega |s| \cdot |f(x)|\, dx = |s| \int_\Omega |f(x)|\, dx = |s|\|f\|_1.$$

Note that $|s|$ denotes the modulus.
Hence, the absolute homogeneity property is satisfied.

Positive definiteness/positiveness: For a real valued function f, we have $|f(x)| \geq 0$ and $|f(x)| = 0$ means $x = 0$. So $\int_\Omega |f(x)| dx \ \geq 0$.

Let $f \in L^1(\Omega)$ such that $\|f\|_1 = 0$. This implies that $\int_\Omega |f(x)|\, dx = 0$. Since $|f(x)| \geq 0$ for all $x \in \Omega$, it follows that $|f(x)| = 0$ almost everywhere on Ω. Consequently, $f(x) = 0$ almost everywhere on Ω. Therefore, if $\|f\|_1 = 0$, then $f = 0$ in $L^1(\Omega)$. Thus, the positive definiteness property holds.

Since all three properties are satisfied, we conclude that the map $f \mapsto \|f\|_1$ defines a norm on $L^1(\Omega)$. □