Reminder XIII

- $(A, D(A))$ <u>self-adjoint</u> if $(A, D(A)) = (A^*, D(A^*))$.

  If so, $\langle Af, g \rangle = \langle f, Ag \rangle$ $\forall f, g \in D(A)$ but this does not define self-adjointness, only that A is <u>symmetric</u>.

- $(A, D(A))$ <u>invertible</u> if $\ker(A) = \{0\}$, and then $(A^{-1}, D(A^{-1}))$ exists with $D(A^{-1}) = \operatorname{Ran}(A)$. If $\operatorname{Ran}(A) = \mathcal{H}$, then $A^{-1} \in \mathcal{B}(\mathcal{H})$. boundedly invertible operator

- <u>Resolvent set</u> $\rho(A) := \{ z \in \mathbb{C} \mid (A - z\mathbb{1})^{-1} \in \mathcal{B}(\mathcal{H}) \}$
  open set in $\mathbb{C}$

  <u>Spectrum</u> $\sigma(A) := \mathbb{C} \setminus \rho(A) = \{ z \in \mathbb{C} \mid$ either
  closed set in $\mathbb{C}$

  $\ker(A - z\mathbb{1}) \neq \{0\}$, or $\ker(A - z\mathbb{1}) = \{0\}$ but $(A - z\mathbb{1})^{-1} \notin \mathcal{B}(\mathcal{H}) \}$

- <u>Example</u>: if $\varphi \in C_b(\mathbb{R}^n)$, and M_φ = multiplication op.
  closure in $\mathbb{C}$
  in $L^2(\mathbb{R}^n)$, then $\sigma(M_\varphi) = \overline{\varphi(\mathbb{R}^n)}$

  eigenvector or eigenfunction
- $z \in \mathbb{C}$ is an <u>eigenvalue</u> of A if $\exists f \in D(A)$, $f \neq 0$ s.t. $Af = zf$. Set of all eigenvalue: $\underline{\sigma_p(A)} \subset \sigma(A)$

- <u>Thm</u>: If $(A, D(A))$ is self-adjoint, $\sigma(A) \subset \mathbb{R}$.
  2 eigenvectors associated with 2 different eigenvalues are orthogonal.

Examples of self-adjoint operators :

1) Multiplication operator in $L^2(\mathbb{R}^n)$:

$\varphi : \mathbb{R}^n \to \mathbb{R}$, measurable. in order to be self-adjoint

$M_\varphi \equiv \varphi(X) \equiv \varphi(Q)$ with $[M_\varphi f](x) = \varphi(x) f(x)$

for $f \in D(M_\varphi) = \{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |\varphi(x) f(x)|^2 dx < \infty \}$.

Examples $\varphi(X) = X_j$, $\varphi(X) = (1 + X^2)^{s/2} =: \langle X \rangle^s$

with $D(\langle X \rangle^s) = \mathcal{H}_s(\mathbb{R}^n)$ weighted Hilbert space