Reminder X

contain its limit points

- Subspace, closed subspace,
- Examples: $\mathrm{Vect}(f_1, \ldots, f_n)$ closed subspace

  $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, subspace, not closed

they are dense subspaces

  $\mathcal{M}^\perp := \{ f \in \mathcal{H} \mid \langle f, g \rangle = 0 \ \forall g \in \mathcal{M} \}$

  closed subspace for any set $\mathcal{M}$.
- $\mathcal{M}$ is dense in $\mathcal{H}$ if and only if $\mathcal{M}^\perp = \{0\}$.
- dual = set of continuous linear functionals

  $\mathcal{H}^* \simeq \mathcal{H}$, Riesz lemma.

set of all bounded linear operators

- Bounded linear operators, $\mathcal{B}(\mathcal{H})$

  ↳ norm $\|B\| = \sup_{\substack{f \in \mathcal{H} \\ f \neq 0}} \dfrac{\|Bf\|}{\|f\|} = \inf\{ c \geq 0 \mid \|Bf\| \leq c\|f\| \}$

  $= \sup_{\substack{f \in \mathcal{H}_1 \\ \|f\| = 1}} \sup_{\substack{g \in \mathcal{H}_2 \\ \|g\| = 1}} |\langle f, Bg \rangle|$.

$\mathcal{B}(\mathcal{H})$ is a normed algebra: $\|AB\| \leq \|A\| \, \|B\|$.

Examples: $M_n(\mathbb{C})$ if $\mathcal{H} = \mathbb{C}^n$, multiplication operator for $\mathcal{H} = L^2(\mathbb{R}^n)$.

- <u>Adjoint</u> : $\forall B \in \mathcal{B}(\mathcal{H})$, $\exists B^* \in \mathcal{B}(\mathcal{H})$ s.t.

$$\langle B^* f, g \rangle = \langle f, Bg \rangle \quad \forall f, g \in \mathcal{H}.$$

<u>Properties</u> : $\|B^*\| = \|B\| = \|B^* B\|^{1/2}$.

$$(B^*)^* = B$$

$$(AB)^* = B^* A^*$$

- 3 main <u>convergences</u> in $\mathcal{B}(\mathcal{H})$:

uniform, strong, weak