Spectrum and Eigenvalues of Bounded Multiplication Operators

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Let M_φ be a multiplication operator on $L^2(\mathbb{R})$ such that $M_\varphi f = \varphi f$. If φ is an essentially bounded function, in short $\varphi \in L^\infty(\mathbb{R})$, then M_φ is also bounded. This is because the operator norm of M_φ equals to the essential supremum of $|\varphi|$. In this report, let us characterize the spectrum and the eigenvalues of multiplication operators and find the multiplicity of an eigenvector.

First of all, if φ is a continuous function then $\sigma(M_\varphi)$ becomes $\overline{\operatorname{Ran}\varphi} \subset \mathbb{C}$, where $\operatorname{Ran}\varphi$ denotes the range of φ and $\overline{\operatorname{Ran}\varphi}$ denotes the closure of $\operatorname{Ran}\varphi$. If λ does not belong to $\overline{\operatorname{Ran}\varphi}$, then there exists an $\epsilon > 0$ such that $\inf_{x\in\mathbb{R}} |\varphi(x) - \lambda| \geq \epsilon$. This implies $\sup_{x\in\mathbb{R}} |\varphi(x) - \lambda|^{-1} \leq \epsilon^{-1}$. Since $\psi := (\varphi - \lambda)^{-1}$ is a bounded function, the bounded operator M_ψ can be defined and M_ψ is the inverse of $M_\varphi - \lambda I$. Hence, λ does not belong to $\sigma(M_\varphi)$. Thus, $\sigma(M_\varphi) \subset \overline{\operatorname{Ran}\varphi}$. On the other hand, supposed that $\lambda \in \operatorname{Ran}\varphi$, then for all natural numbers n, $B(\lambda, 1/n)$ denote the open ball around λ with radius $1/n$ and let E_n be the inverse image of $B(\lambda, 1/n)$ by φ. By continuity of φ, each E_n is an open set of $\mathbb{R}$ and not empty, and if necessary E_n can be regarded as of finite measure. This is because there exists at least one of $j \in \mathbb{Z}$ such that $E_n \cap [j, j+1)$ is not of measure zero, and consider such an $E_n \cap [j, j+1)$ instead of E_n. Thus, the indicator function 1_{E_n} belongs to $L^2(\mathbb{R})$ and is not zero. This leads the following estimate:

$$\|(M_\varphi - \lambda I)1_{E_n}\|^2 = \int |(\varphi(x) - \lambda)1_{E_n}(x)|^2 dx \leq \sup_{x\in E_n} |\varphi(x) - \lambda|^2 \|1_{E_n}\|^2 \leq \frac{1}{n^2}\|1_{E_n}\|^2.$$

Suppose now that λ does not belong to $\sigma(M_\varphi)$, then there exists a bounded inverse for $M_\varphi - \lambda I$. Meanwhile, the above estimate implies the following result.

$$\|(M_\varphi - \lambda I)^{-1}\| \geq \frac{\|(M_\varphi - \lambda I)^{-1}(M_\varphi - \lambda I)1_{E_n}\|}{\|(M_\varphi - \lambda I)1_{E_n}\|} = \frac{\|1_{E_n}\|}{\|(M_\varphi - \lambda I)1_{E_n}\|} \geq n.$$

This holds for all natural numbers n and this contradict is the boundedness of $(M_\varphi - \lambda I)^{-1}$. Therefore, λ is contained in $\sigma(M_\varphi)$. Since $\operatorname{Ran}\varphi \subset \sigma(M_\varphi)$ and $\sigma(M_\varphi)$ is closed, $\overline{\operatorname{Ran}\varphi} \subset \sigma(M_\varphi)$. $\sigma(M_\varphi) = \overline{\operatorname{Ran}\varphi}$ has been shown.

When $\varphi \in L^\infty(\mathbb{R})$, $\lambda \in \sigma(M_\varphi)$ if and only if for any $\epsilon > 0$ the measure of $\varphi^{-1}(B(\lambda, \epsilon))$ is not zero, where $\varphi^{-1}(B(\lambda, \epsilon))$ denotes the inverse image of the set $B(\lambda, \epsilon) \subset \mathbb{C}$ by φ. If there exists an $\epsilon > 0$ such that $\varphi^{-1}(B(\lambda, \epsilon))$ is of measure zero, then since difference on measure zero can be ignored as $L^\infty(\mathbb{R})$, a bounded inverse $M_{(\varphi-\lambda)^{-1}}$ can be defined, whose norm is $1/\epsilon$ or less. On the other hand, if for any $\epsilon > 0$ the measure of $\varphi^{-1}(B(\lambda, \epsilon))$ is not zero and suppose that $M_\varphi - \lambda$ has a bounded inverse, then this leads to a contradiction in the same way as the case when φ is continuous.

Let us think about the eigenvalue of M_φ. $\lambda \in \sigma_p(M_\varphi)$ if and only if $\varphi^{-1}(\{\lambda\})$ is not of measure zero. Indeed, if $\varphi^{-1}(\{\lambda\})$ is not of measure zero, then there exists at least one of $j \in \mathbb{Z}$ such that $\varphi^{-1}(\{\lambda\}) \cap [j, j+1)$ is not of measure zero. Let F_λ be such a $\varphi^{-1}(\{\lambda\}) \cap [j, j+1)$, then the indicator function is an eigenvector of M_φ with an eigenvalue λ. On the other hand, if λ is an eigenvalue of M_φ then there exists the eigenvector f and the support of f is included in $\varphi^{-1}(\{\lambda\})$. Thus, $\varphi^{-1}(\{\lambda\})$ is not of measure zero.

As for the multiplicity, any eigenvalue has infinite multiplicity. It will be shown that for any natural number n, an eigenvalue has n linearly independent eigenvectors. Let λ be an eigenvalue, and then the above result implies $\varphi^{-1}(\{\lambda\})$ is not of measure zero. There exists at least one of $j \in \mathbb{Z}$ such that $\varphi^{-1}(\{\lambda\}) \cap [j, j+1)$ is not of measure zero. Let G_λ be such a $\varphi^{-1}(\{\lambda\}) \cap [j, j+1)$, Let $\alpha := \mu(G_\lambda)$, where $\mu(G_\lambda)$ denotes the measure of G_λ, and $I_j := [\alpha j/n, \alpha(j+1)/n)$ for any integer j. Since $\alpha = \sum_{j \in \mathbb{Z}} \mu(G_\lambda \cap I_j)$ and $0 \leq \mu(G_\lambda \cap I_j) \leq \alpha/n$ for any integer j, at least n of j's, $G_\lambda \cap I_j$ is not of measure zero. These j's are denoted by $j_1, j_2, \ldots, j_n$. Since $G_\lambda \cap I_j$'s are mutually disjoint, the indicator functions are linearly independent and they are n linearly independent eigenvectors.