C^*-ALGEBRAIC METHODS IN SPECTRAL THEORY REPORT

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Exercise 6.1.1 In case $f(x,\xi) = f(\xi)$. We have

$$\begin{aligned}
[\mathfrak{Op}(f)u](x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(x-y)\cdot\eta} f(\eta)u(y)dyd\eta \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\widehat{\mathbb{R}^d}} e^{ix\eta} f(\eta)\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-iy\eta}u(y)dyd\eta \\
&= [\mathcal{F}^* f(X)\mathcal{F}u](x) \\
&= [f(D)u](x).
\end{aligned}$$

In case $f(x,\xi) = f(x)$, we have

$$\begin{aligned}
[\mathfrak{Op}(f)u](x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(x-y)\cdot\eta} f\left(\frac{x+y}{2}\right)u(y)dyd\eta \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\eta}\hat{g_x}(\eta)d\eta \\
&= [\mathcal{F}^*\mathcal{F}g_x](x) \\
&= g_x(x) \\
&= f(x)u(x) \\
&= [f(X)u](x)
\end{aligned}$$

where $g_x(y) = f\left(\frac{x+y}{2}\right)u(y)$.

Exercise 6.1.2 For $u \in \mathcal{H}$, we have

$$\begin{aligned}
[\mathfrak{Op}(f)u](z) &= \frac{1}{(2\pi)^d}\int_{\Xi}[(\mathcal{F}_\Xi^{-1}f)(\mathsf{x})\cdot W(\mathsf{x})u](z)d\mathsf{x} \\
&= \frac{1}{(2\pi)^d}\int_{\Xi}(\mathcal{F}_\Xi^{-1}f)(\mathsf{x})e^{-i(\frac{x}{2}+z)\xi}u(x+z)d\mathsf{x} \\
&= \frac{1}{(2\pi)^{2d}}\int_{\Xi}\int_{\Xi} e^{i(y\xi-x\eta)}f(\mathsf{y})e^{-i(\frac{x}{2}+z)\xi}u(z+x)d\mathsf{x}d\mathsf{y} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\Xi}\delta\left(y-\frac{x}{2}-z\right)f(\mathsf{y})e^{-ix\eta}u(x+z)dxd\mathsf{y} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} f\left(\frac{x'+z}{2},\eta\right)e^{-i(x'-z)\eta}u(x')dx'd\eta
\end{aligned}$$

where $x' = x + z$, which coincides with (6.1.1).

$$\begin{aligned}\mathfrak{Op}(f \circ g) &= \frac{1}{(2\pi)^d}\int_\Xi (\mathcal{F}_\Xi^{-1}(f\circ g))(\mathsf{x})W(\mathsf{x})d\mathsf{x}\\ &= \frac{1}{(2\pi)^d}\int_\Xi \frac{1}{(2\pi)^d}\int_\Xi e^{i\sigma(\mathsf{x},\mathsf{y})}(f\circ g)(\mathsf{y})d\mathsf{y}W(\mathsf{x})d\mathsf{x}\\ &= \frac{1}{(2\pi)^{2d}}\int_\Xi\int_\Xi e^{i\sigma(\mathsf{x},\mathsf{y})}\frac{4^d}{(2\pi)^{2d}}\int_\Xi\int_\Xi e^{-2i\sigma(\mathsf{y}-\mathsf{z},\mathsf{y}-\mathsf{w})}f(\mathsf{z})g(\mathsf{w})W(\mathsf{x})d\mathsf{z}d\mathsf{w}d\mathsf{y}d\mathsf{x}\\ &= \frac{4^d}{(2\pi)^{4d}}\int_\Xi\int_\Xi\int_\Xi\int_\Xi e^{i\sigma(\mathsf{x},\mathsf{y})}e^{-2i\sigma(\mathsf{y}-\mathsf{z},\mathsf{y}-\mathsf{w})}f(\mathsf{z})g(\mathsf{w})W(\mathsf{x})d\mathsf{z}d\mathsf{w}d\mathsf{y}d\mathsf{x}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathfrak{Op}(f)\mathfrak{Op}(g) &= \frac{1}{(2\pi)^d}\int_\Xi(\mathcal{F}_\Xi^{-1}f)(\mathsf{x})W(\mathsf{x})d\mathsf{x}\frac{1}{(2\pi)^d}\int_\Xi(\mathcal{F}_\Xi^{-1}g)(\mathsf{y})W(\mathsf{y})d\mathsf{y}\\ &= \frac{1}{(2\pi)^{4d}}\int_\Xi\int_\Xi e^{i\sigma(\mathsf{x},\mathsf{z})}f(\mathsf{z})d\mathsf{z}W(\mathsf{x})d\mathsf{x}\int_\Xi\int_\Xi e^{i\sigma(\mathsf{y},\mathsf{w})}g(\mathsf{w})d\mathsf{w}W(\mathsf{y})d\mathsf{y}\\ &= \frac{1}{(2\pi)^{4d}}\int_\Xi\int_\Xi\int_\Xi\int_\Xi e^{i\sigma(\mathsf{x},\mathsf{z})}e^{i\sigma(\mathsf{y},\mathsf{w})}f(\mathsf{z})g(\mathsf{w})W(\mathsf{x})W(\mathsf{y})d\mathsf{z}d\mathsf{x}d\mathsf{w}d\mathsf{y}\\ &= \frac{4^d}{(2\pi)^{4d}}\int_\Xi\int_\Xi\int_\Xi\int_\Xi e^{i\sigma(\mathsf{x}'-2\mathsf{y}',\mathsf{z})}e^{2i\sigma(\mathsf{y}',\mathsf{w})}f(\mathsf{z})g(\mathsf{w})e^{i\sigma(\mathsf{x}'-2\mathsf{y}',\mathsf{y}')}W(\mathsf{x}')d\mathsf{x}'d\mathsf{y}'d\mathsf{z}d\mathsf{w}\end{aligned}$$

where $\mathsf{x}' = \mathsf{x}+\mathsf{y}$, $\mathsf{y}' = \frac{\mathsf{y}}{2}$ (the absolute value of corresponding Jacobian is 4^d), and we used the relation $W(\mathsf{x})W(\mathsf{y}) = e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}W(\mathsf{x}+\mathsf{y})$. Moreover, we also have

$$\begin{aligned}e^{i\sigma(\mathsf{x}'-2\mathsf{y}',\mathsf{z})}e^{2i\sigma(\mathsf{y}',\mathsf{w})}e^{i\sigma(\mathsf{x}'-2\mathsf{y}',\mathsf{y}')} &= e^{i\sigma(\mathsf{x}'-2\mathsf{y}',\mathsf{z}+\mathsf{y}')}e^{2i\sigma(\mathsf{y}',\mathsf{w})}\\ &= e^{i\sigma(\mathsf{x}',\mathsf{z}+\mathsf{y}')+i\sigma(-2\mathsf{y}',\mathsf{z}+\mathsf{y}')}e^{2i\sigma(\mathsf{y}',\mathsf{w})}\\ &= e^{i\sigma(\mathsf{x}',\mathsf{z}+\mathsf{y}')}e^{-2i\sigma(\mathsf{y}',\mathsf{z}+\mathsf{y}'-\mathsf{w})}\\ &= e^{i\sigma(\mathsf{x}',\mathsf{y}'')}e^{-2i\sigma(\mathsf{y}''-\mathsf{z},\mathsf{y}''-\mathsf{w})}\end{aligned}$$

where $\mathsf{y}'' = \mathsf{y}'+\mathsf{z}$. So we get

$$\begin{aligned}\mathfrak{Op}(f)\mathfrak{Op}(g) &= \frac{4^d}{(2\pi)^{4d}}\int_\Xi\int_\Xi\int_\Xi\int_\Xi e^{i\sigma(\mathsf{x}',\mathsf{y}'')}e^{-2i\sigma(\mathsf{y}''-\mathsf{z},\mathsf{y}''-\mathsf{w})}f(\mathsf{z})g(\mathsf{w})W(\mathsf{x}')d\mathsf{x}'d\mathsf{y}''d\mathsf{z}d\mathsf{w}\\ &= \mathfrak{Op}(f\circ g).\end{aligned}$$

Exercise 6.2.1

$$\begin{aligned}(f \circ_{\frac{1}{2}\mathbf{1}}^{1} g)(x;\xi) &= \int_G\int_G\int_{\hat G}\int_{\hat G}\xi(y)\overline{\eta(z)\zeta(y-z)}f\left(x+\frac{1}{2}(z-y);\eta\right)g\left(x+\frac{z}{2};\zeta\right)dydzd\eta d\zeta\\ &= 4^d\int_G\int_G\int_{\hat G}\int_{\hat G}\xi(2(z'-y'))\overline{\eta(2z'-2x)\zeta(2x-2y')}f(y',\eta)g(z',\zeta)dy'dz'd\eta d\zeta\end{aligned}$$

where $y' = x+\frac{1}{2}(z-y), z' = x+\frac{z}{2}$. Note that $\overline{\xi(x)}$ corresponds to $e^{-ix\xi}$ (x,ξ in the index are elements of $\mathbb{R}^d, \hat{\mathbb{R}}^d$). So $\xi(2(z'-y'))\overline{\eta(2z'-2x)\zeta(2x-2y')}$ corresponds to $e^{2i(z'-y')\xi}e^{-2i(z'-x)\eta}e^{-2i(x-y')\zeta}$. Recall that the Moyal product is defined as follows:

$$(f \circ g)(\mathsf{x}) = \frac{4^d}{(2\pi)^d}\int_\Xi\int_\Xi e^{-2i\sigma(\mathsf{x}-\mathsf{y},\mathsf{x}-\mathsf{z})}f(\mathsf{y})g(\mathsf{z})d\mathsf{y}d\mathsf{z}$$

To see the correspondence between $f \circ^1_{\frac{1}{2}\mathbf{1}} g$ and $f \circ g$, we need to see $e^{-2i\sigma(\mathsf{x}-\mathsf{y},\mathsf{x}-\mathsf{z})} = e^{-2i\{(y-z)\xi+(z-x)\eta+(x-y)\zeta)\}}$ for $\mathsf{x},\mathsf{y},\mathsf{z} \in \Xi$, which is confirmed by the following computation.

$$\begin{aligned}
e^{-2i\sigma(\mathsf{x}-\mathsf{y},\mathsf{x}-\mathsf{z})} &= e^{-2i\{\sigma(\mathsf{x},\mathsf{x})-\sigma(\mathsf{x},\mathsf{z})-\sigma(\mathsf{y},\mathsf{x})+\sigma(\mathsf{y},\mathsf{z})\}} \\
&= e^{-2i\{-(z\xi-x\zeta)-(x\eta-y\xi)+(z\eta-y\zeta)\}} \\
&= e^{-2i\{(y-z)\xi+(z-x)\eta+(x-y)\zeta\}}.
\end{aligned}$$

As for the involution, we have

$$(f^{\circ^1_{\frac{1}{2}\mathbf{1}}})(x;\xi) = \int_G\int_{\hat{G}}[\xi\eta^{-1}](y)\overline{f(x;\eta)}dyd\eta.$$

This integral corresponds to

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int_\Xi e^{iy(\xi-\eta)}\overline{f(x,\eta)}dyd\eta &= \int_{\mathbb{R}^d}\delta(\xi-\eta)\overline{f(x,\eta)}d\eta \\
&= \overline{f(x,\xi)} \\
&= f^\circ(x,\xi)
\end{aligned}$$

where $(x,\xi) \in \Xi$.

Exercise 7.2.1 By the following computation, we get the relation $U^A(y)V_\xi = e^{-iy\xi}V_\xi U^A(y)$.

$$[V_\xi U^A(y)u](x) = e^{-ix\xi}[U^A(y)u](x) = e^{-ix\xi}e^{-iy\int_0^1 A(x+sy)ds}u(x+y)$$

$$[U^A(y)V_\xi u](x) = e^{-iy\int_0^1 A(x+sy)ds}[V_\xi u](x+y) = e^{-iy\int_0^1 A(x+sy)ds}e^{-i(x+y)\xi}u(x+y)$$

Combining this with the relation $U^A(y)U^A(z) = \pi[\omega^B(y,z)]U^A(y+z)$ we get

$$\begin{aligned}
W^A(\mathsf{x})W^A(\mathsf{y}) &= e^{-\frac{i}{2}x\xi}V_\xi U^A(x)e^{-\frac{i}{2}y\eta}V_\eta U^A(y) \\
&= e^{-\frac{i}{2}(x\xi+y\eta)}V_\xi U^A(x)V_\eta U^A(y) \\
&= e^{-\frac{i}{2}(x\xi+y\eta)}V_\xi U^A(x)e^{iy\eta}U^A(y)V_\eta \\
&= e^{-\frac{i}{2}(x\xi-y\eta)}V_\xi\pi[\omega^B(x,y)]U^A(x+y)V_\eta \\
&= e^{-\frac{i}{2}(x\xi-y\eta)}\pi[\omega^B(x,y)]V_\xi e^{-i(x+y)\eta}V_\eta U^A(x+y) \\
&= e^{-\frac{i}{2}(x\xi-y\eta)}e^{-i(x+y)\eta}e^{\frac{i}{2}(x+y)(\xi+\eta)}\pi[\omega^B(x,y)]e^{-\frac{i}{2}(x+y)(\xi+\eta)}V_{\xi+\eta}U^A(x+y) \\
&= e^{\frac{i}{2}(-x\eta+y\xi)}\pi[\omega^B(x,y)]W^A(\mathsf{x}+\mathsf{y}) \\
&= e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\pi[\omega^B(x,y)]W^A(\mathsf{x}+\mathsf{y}).
\end{aligned}$$