G30 Final Report

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1. Introduction

After reading Section 3.2 of the book [RC], I decide to summarize how to evaluate π using the Monte Carlo method.

2. Monte Carlo approximation

f is a measurable function and X_i, $i=1,\cdots,M$ is independent random variable with identical PDF π_X. Then the Monte Carlo approximation to $\mathbb{E}[f(x)]$ is given by

$$\bar{f}_M = \sum_{i=1}^{M} w_i f(x_i),$$

where $(x_1, x_2, \cdots, x_M) = (X_1(\omega), X_2(\omega), \cdots, X_M(\omega))$ are the i.i.d. samples and $w_i = \frac{1}{M}$ denote uniform weights.

3. how to evaluate π using the Monte Carlo method

First, we consider randomly hitting a certain dots in a 1×1 square as shown in Figure 1 and count up the number of dots struck in the shaded area among them.

Let n be the number of dots struck within the 1×1 square and m be the number of dots within the shaded area of the square. Then, π is given by

$$\frac{\pi}{4} = \frac{m}{n} \qquad \therefore \pi = \frac{4m}{n}.$$

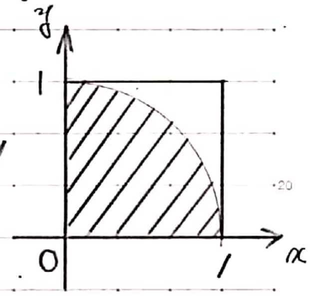

Figure 1

Second, we consider some examples.

Example 3-1

We consider 17 dots randomly hit as shown in Figure 2. In this case, $n=17$, and $m=13$. Then, π is given by

$$\pi = \frac{4\times13}{17} = \frac{52}{17} = 3.0588\cdots\cdots.$$

Therefore, the error is about 3 percent.

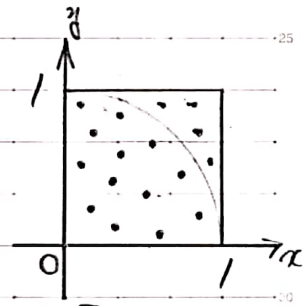

Figure 2

Example 3-2

We consider 50 dots randomly hit as shown in Figure 3. In this case, $n=50$, and $m=39$. Then π is given by

$\pi = \frac{4 \times 39}{50} = \frac{78}{25} = 3.12$

Therefore, the error is about 0.6 percent.

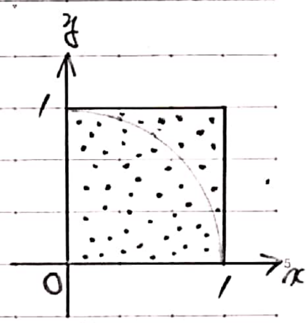

Figure 3

From the two examples, it appears that the accuracy increases as n increases. Thus, the next step is to consider the relationship between n and the accuracy of this method.

4. Chebyshev's inequality (Chernoff Bound)

Let $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ with probability p_i, and $X_i = 0$ with probability $1 - p_i$, and all X_i are independent. Let $\mu = \mathbb{E}(x) = \frac{1}{n}(\sum_{i=1}^{n} p_i)$

$\mathbb{P}(X \geq (1+\delta)\mu) \leq \exp(-\frac{\delta^2}{2+\delta}\mu)$ for all $\delta > 0$ (Upper Tail)

$\mathbb{P}(X \leq (1-\delta)\mu) \leq \exp(-\frac{\mu\delta^2}{2})$ for all $0 < \delta < 1$ (Lower Tail)

For $\delta \in (0, 1)$, we can combine the lower and upper tails to obtain the following bound.

$\mathbb{P}(|X - \mu| \geq \delta\mu) \leq 2\exp(-\frac{\mu\delta^2}{3})$ for all $0 < \delta < 1$ ($\mu = \mathbb{E}(x)$)

5. Relationship between n and the accuracy of Monte Carlo method

By using Chernoff Bound, we can see that $\pi(1-\delta) < \frac{4m}{n} < \pi(1+\delta)$ will be satisfied with probability greater than $1 - 2\exp(-\frac{\pi n \delta^2}{12})$.

We consider one specific example.

Example 5-1

Q. How many times do you have to hit the dot in order to get to within a 3 percent margin of error, with a probability of 80 percent or better?

A. In this case, $\delta = 0.03$. Then n is given by

$$1 - 2\exp(-\frac{\pi n \cdot 0.03^2}{12}) \geq 0.8.$$

By calculating this inequality, we can see that we need to hit the dot approximately 10,000 times.

6. Conclusion

Having written this report, I now understand that the Monte Carlo method is very useful.

7. Reference lists

doc.gijodai.ac.jp/vm/cli-pro/ITK01/pai.PDF
https://manabitimes.jp/math/1182
https://math.mit.edu/~goemans/18310S15/chernoff-notes.pdf