report

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Exercise 2.4.16

We write $\Phi : \mathbf{\Omega} \ni x \mapsto \tau_x \in \Omega(C(\mathbf{\Omega}))$, and we prove that Φ is a homeomorphism.

Φ is continuous:

Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net converging a point $x \in \Omega$. Since the topology of $\Omega(C(\mathbf{\Omega}))$ is weak* topology, we have $\tau_{x_\lambda} \to \tau_x$ since for all $f \in C(\mathbf{\Omega})$, $f(x_\lambda) \to f(x)$. So Φ is continuous.

Φ is bijective:

If $x_1, x_2 \in \Omega$ are distinct point, by Urisohn's lemma there is a function $f \in C(\Omega)$ such that $f(x_1) = 0$ and $f(x_2) = 1$. Therefore $\Phi(x_1) \neq \Phi(x_2)$. To show surjectivity of Φ, we take $\tau \in \Omega(C(\mathbf{\Omega}))$. Then $\ker(\tau)$ become a proper C* subalgebra of $C(\mathbf{\Omega})$, and $\ker(\tau)$ separates points of $\mathbf{\Omega}$. Because we can find some function $f \in C(\mathbf{\Omega})$ such that $f(x_1) \neq f(x_2)$ for some distinct point $x_1, x_2 \in \mathbf{\Omega}$. And we take $g = f - \tau(f)$, g belongs to $\ker(\tau)$ and $g(x_1) \neq g(x_2)$. By Stone-Weierstrass Theorem, there is a point $x \in \mathbf{\Omega}$ such that $f(x) = 0$ for all $f \in \ker(\tau)$. Applying this for g, we have

$$0 = (f - \tau(f))(x) = f(x) - \tau(f)(x) = f(x) - \tau(f).$$

So, $f(x) = \tau(f)$ for all $f \in C(\mathbf{\Omega})$ i.e. $\tau = \Phi(x)$.

As a result, we prove that Φ is continuous and bijective on $\mathbf{\Omega}$. Since $\mathbf{\Omega}$ is compact, $\Phi(\mathbf{\Omega})$ is compact and for any closed subset $F \subset \Omega$, $(\Phi^{-1})^{-1}(F) = \Phi(F)$ is closed because Ω is Hausdorff and Φ is continuous. Therefore Φ^{-1} is also continuous.

□

Exercise 3.1.12

G: locally compact group, $f \in C_c(G), x \in G$, μ: left Haar measure on G. By the definition of the modular function, we have

$$\mu(Ex) = \Delta(x)\mu(E)$$

for any Borel set $E \subset G$, and by the definition of the left Haar measure, we also have

$$\mu(xE) = \mu(E).$$

So, by translating y to $x^{-1}y$ we have

$$\int_G f(xy)\, d\mu(y) = \int_G f(y)\, d\mu(y).$$

And by translating y to yx^{-1}, the measure μ is changed

$$\mu(Ex^{-1}) = \Delta(x^{-1})\mu(E).$$

Since the modular function Δ is a homomorphism, we have

$$\int_G f(yx)\, d\mu(y) = \int_G f(y)\Delta(x^{-1})\, d\mu(y) = \Delta(x)^{-1} \int_G f(y)\, d\mu(y).$$

Finally, if we define $\lambda(E) := \mu(E^{-1})$, then λ is a right Haar measure and $d\lambda(x) = \Delta(x^{-1})d\mu(x)$. So $d\mu(x^{-1}) = \Delta(x^{-1})d\lambda(x^{-1}) = \Delta(x^{-1})d\mu(x)$, and therefore we have

$$\int_G \Delta(y^{-1})f(y^{-1})\, d\mu(y) = \int_G f(y^{-1})\, d\mu(y^{-1}) = \int_G f(y)\, d\mu(y).$$

□