

# Irrational rotation algebra

Definition (rational, irrational rotation algebra)

$$\mathcal{C} := C(\mathbb{T}), \quad G := \mathbb{Z}, \quad [O_\theta f](z) := f(e^{i2\pi\theta} z) \quad (f \in \mathcal{C}, z \in \mathbb{T})$$

$\theta \in [0, 1]$

$$A_\theta := C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$$

If  $\theta$  is irrational/rational,  $A_\theta$  is called irrational/rational algebra.

In this report, we prove that universality of  $A_\theta$  and simplicity of  $A_\theta$ . Next, we introduce conditional expectation which is used to prove simplicity of  $A_\theta$  and other  $C^*$ -algebras. Lastly we introduce some examples of irrational rotation algebra and some classification theorems without proof.

First, we prove that  $A_\theta$  is the universal  $C^*$ -algebra generated by two unitaries  $u, v$  satisfying  $uv = pvu$ .  
( $p = e^{2\pi i\theta}$ )

Lemma 1.

$A_\theta$  is generated by two unitaries  $u, v$  such that  $uv = pvu$  ( $p = e^{2\pi i\theta}$ ).

Proof).

$A_\theta$  is completion of  $*$ -algebra  $C_c(\mathbb{Z} \times \mathbb{T})$ .

• Convolution product

$$\begin{aligned} f * g(n, z) &= \int_{\mathbb{T}} f(m, z) O_m g(n-m, z) dm \\ &= \sum_{m=-\infty}^{\infty} f(m, z) g(n-m, p^{-m} z) \end{aligned}$$

• involution

$$\begin{aligned} f^*(n, z) &= \Delta(n)^{-1} \overline{f(-n, z)} \\ &= \overline{f(-n, p^n z)} \end{aligned}$$



Notation.

$$\varphi \otimes h(n, z) := \varphi(z) h(n) \quad (\varphi \in C(\mathbb{T}), h \in C(\mathbb{Z}))$$

$$\delta_n(m) := \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

$$i_{C(\mathbb{T})}(\varphi) := \varphi \otimes \delta_0$$

$$L_{\mathbb{T}}(z) := z$$

then  $1 \otimes \delta_0$  is identity and  $u := 1 \otimes \delta_1, v := L_{\mathbb{T}} \otimes \delta_0$  are unitaries in  $A_0$ .

$$(\textcircled{1}) \cdot (1 \otimes \delta_0) * f(n, z) = \sum_{m=-\infty}^{\infty} 1(z) \delta_0(m) \cdot f(n-m, p^{-m}z) = f(n, z)$$

$$\cdot (1 \otimes \delta_1)^* (1 \otimes \delta_1) = \sum_{m=-\infty}^{\infty} |p^{-m}z| \delta_1(-m) \cdot |p^{-m}z| \delta_1(n-m) = 1 \otimes \delta_0$$

And

$$\cdot i_{C(\mathbb{T})}(\varphi) * u^n = (\varphi \otimes \delta_0) * (1 \otimes \delta_1)^n = \varphi \otimes \delta_0$$

$$\cdot u * i_{C(\mathbb{T})}(\varphi)^* u^* = (1 \otimes \delta_1) * (\varphi \otimes \delta_0)^* (1 \otimes \delta_1)^*$$

$$= (0, \varphi \otimes \delta_1) * (1 \otimes \delta_{-1})$$

$$= 0, \varphi \otimes \delta_0$$

$$= i_{C(\mathbb{T})}(0, \varphi)$$

$\text{span} \{ \varphi \otimes \delta_n \mid \varphi \in C(\mathbb{T}), n \in \mathbb{Z} \}$  is dense in  $C_c(\mathbb{Z} \times \mathbb{T})$ ,  
( $\textcircled{1}$  [1] lemma 1.87)

and  $\text{span} \{ v^n \mid n \in \mathbb{Z} \}$  is dense in  $i_{C(\mathbb{T})}(C(\mathbb{T}))$ .

( $\textcircled{2}$ )  $v = L_{\mathbb{T}} \otimes \delta_0, L_{\mathbb{T}}(z) = z$ . The Stone-Weierstrass implies  $L_{\mathbb{T}}$  generates  $C(\mathbb{T})$ , thus  $\text{span} \{ v^n \mid n \in \mathbb{Z} \}$  is dense in  $i_{C(\mathbb{T})}(C(\mathbb{T}))$ .

Therefore,  $u, v$  generates  $A_0$ .

and  $uv = puu$ .

$$\left( \begin{aligned} \textcircled{1} u * v(n, z) &= \sum_{m=-\infty}^{\infty} 1 \cdot \delta_1(m) p^{-m} z p_0(n-m) = p^{-1} L_{\mathbb{T}} \otimes \delta_1(n, z) \\ p u * u(n, z) &= \sum_{m=-\infty}^{\infty} z \cdot \delta_0(m) \cdot \delta_1(n-m) = L_{\mathbb{T}} \otimes \delta_1(n, z) \end{aligned} \right).$$



# Proposition

If  $\theta$  is irrational,  $A_\theta$  is the universal  $C^*$ -algebra.

(i.e. If  $\mathcal{H}$  is Hilbert sp,  $U, V \in B(\mathcal{H})$  are unitaries such that  $UV = pVU$ , then there exists  $*$ -homomorphism  $\varphi: A_\theta \rightarrow C^*(U, V)$  such that  $\varphi(u) = U, \varphi(v) = V$  ( $u, v \in A_\theta$  are generator of  $A_\theta$ )

Proof).

Suppose that  $U, V \in B(\mathcal{H})$  are unitaries which satisfy  $UV = pVU$ .

Claim 1.  $\sigma(U), \sigma(V) = \mathbb{T}$ .

(:)  $\lambda \in \sigma(U) \Leftrightarrow \lambda I - U$  is not invertible.

$$V(\lambda I - U)V^* = \lambda I - VUV^*$$

$$= \lambda I - p^{-1}VU^*V^*$$

$$= \lambda I - p^{-1}U$$

$$\therefore \lambda \in \sigma(p^{-1}U)$$

$$\therefore p\lambda \in \sigma(U)$$

In particular,  $p^n \lambda \in \sigma(U)$ .

Since  $\{p^n \lambda \mid n \in \mathbb{Z}\}$  is dense in  $\mathbb{T}$  ( $\theta$  is irrational!) and  $\sigma(U)$  is closed, the claim follows.  $\square$

Thus, spectrum theorem implies  $\pi: C(\mathbb{T}) \rightarrow C^*(U)$

are isomorphism. Let  $U: \mathbb{Z} \rightarrow U(\mathcal{H})$ .

$$\varphi \mapsto \varphi^n$$

$$\text{then } U^n \pi(\varphi) U^{*n} = U^n V U^{*n} = p^n V = \pi(\varphi_n) U$$

Therefore,  $(\pi, U, \mathcal{H})$  is covariant representation of  $(C(\mathbb{T}), \mathbb{Z}, \theta)$

$$\therefore \pi \otimes_{\theta} U(u) = \sum_{m=-\infty}^{\infty} \pi(u(m, \mathbb{Z})) U_m = \pi(1_{C(\mathbb{T})}) U = U$$

$$\pi \otimes_{\theta} U(v) = \sum_{m=-\infty}^{\infty} \pi(v(m, \mathbb{Z})) U_m = \pi(\varphi_{\theta}) V = V$$

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$$u = 1 \otimes \delta_1, v = \varphi \otimes \delta_0$$



It follows that  $\varphi \equiv \pi \otimes U: A_\theta \rightarrow C^*(U, V)$  and

$$\pi \otimes U(u) = U, \quad \pi \otimes U(v) = V. \quad (3)$$

(If  $\theta$  is rational,  $A_\theta$  is also universal. Indeed, we define  $\tau: C(\mathbb{T}) \ni f \mapsto f(\omega) \in C(\mathbb{T})$  and  $\pi := \tilde{\pi} \circ \tau$  ( $\tilde{\pi}: C(\mathbb{T}) \rightarrow C^*(V)$ ), then we obtain homomorphism  $\pi: C(\mathbb{T}) \rightarrow C^*(V)$ .  
 $\uparrow$  Gelfand repn.

Thm If  $\theta$  is irrational,  $A_\theta$  is simple (ie.  $A_\theta$  doesn't have nontrivial ideal)  
 And it follows that  $\pi \otimes U: A_\theta \rightarrow C^*(U, V)$  is isometric  $*$ -isomorphism.

Proof. For each  $\omega \in \mathbb{T}$ , let  $\hat{\tau}_\omega: C_c(\mathbb{Z} \times \mathbb{T}) \rightarrow C_c(\mathbb{Z} \times \mathbb{T})$

be defined  $\hat{\tau}_\omega(f)(n, z) := \omega^n f(n, z)$ . Then,

it is continuous  $*$ -isomorphic. Therefore,

$\hat{\tau}_\omega$  can be extended to an automorphism of  $A_\theta$ .

And,  $\omega \mapsto \hat{\tau}_\omega(a)$  is continuous for all  $a \in A_\theta$ .

Therefore we can define  $\bar{\varphi}: A_\theta \rightarrow A_\theta$  by

$$\bar{\varphi}(a) = \int_{\mathbb{T}} \hat{\tau}_\omega(a) d\omega.$$

Then, it is linear and  $\|\bar{\varphi}\| \leq 1$ . And,

$$\hat{\tau}_\omega(u) = \omega u, \quad \hat{\tau}_\omega(v) = v \quad (\odot \quad u = 1 \otimes \delta_1, \quad v = 1 \otimes \delta_0).$$

$$\hat{\tau}_\omega(v)$$

$$= \hat{\tau}_\omega(1 \otimes \delta_0)$$

$$= \omega^0 (1 \otimes \delta_0)$$

$$= 1 \otimes \delta_0$$

Thus,

$$\begin{aligned} \bar{\varphi}(v^k * u^m) &= \int_{\mathbb{T}} \hat{\tau}_\omega(v^k * u^m) d\omega = v^k * \left( \int_{\mathbb{T}} \omega^m d\omega \right) * u^m \\ &= \begin{cases} v^k & (m=0) \\ 0 & (\text{otherwise}) \end{cases} \quad \dots (*) \end{aligned}$$

Now, Let  $E_n: A_\theta \rightarrow A_\theta$  be defined by

$$E_n(a) := \frac{1}{2n+1} \sum_{j=-n}^n v^j * a * v^{-j}$$

$$\text{Then } E_n(u^k * v^m) = \frac{1}{2n+1} v^k * u^m \sum_{j=-n}^n \omega^{jm} = \begin{cases} v^k & (m=0) \\ 0 & (m \neq 0). \end{cases}$$

( $n$  is sufficiently large)



It follows that

$$\Phi(a) = \lim_{n \rightarrow \infty} E_n(a) \quad a \in \text{span} \{u^k * v^m \mid k, m \in \mathbb{Z}\}$$

by triangular inequality  $\|\Phi(a) - E_n(b)\| \leq \|\Phi(a-b)\| + \|\Phi(b) - E_n(b)\| + \|E_n(b-a)\|$ .

it holds for all  $a \in A_0$ .

Now suppose that  $I$  is a nonzero ideal of  $A$ .

Let  $a \in I$  and  $a \geq 0$ ,  $\rho$  be state on  $A_0$ . Then

$$\rho(\Phi(a)) = \int_{\Pi} \rho(\tilde{\tau}_w(a)) dw > 0 \quad \text{and} \quad \Phi(a) > 0.$$

It follows that ①  $\Phi(a) \in i_{C(\Pi)}(C(\Pi))$ , ②  $\Phi(a) \in I$ .

(\*) ①  $\text{span} \{v^n : n \in \mathbb{Z}\}$  is dense in  $i_{C(\Pi)}(C(\Pi))$ , (\*)  
②  $E_n(a) \in I$  and,  $\lim E_n(a) = \Phi(a)$ .

Thus there exists  $h \in C(\Pi)$  s.t.  $h \not\geq 0$  s.t.  $i_{C(\Pi)}(h) = \Phi(a) \in I$ .

Let  $V$  be neighborhood of  $z_0 \in \Pi$  s.t.  $h(z) > 0 \quad z \in V$ .

Since  $\{p^n z_0 : k \in \mathbb{Z}\}$  is dense in  $\Pi$ , There exists a  $n$

such that  $\bigcup_{k=-n}^n p^k V \supset \Pi$ . Then we define  $g := \sum_{k=-n}^n \theta_k(h)$ .

and  $g(z) > 0$  for all  $z \in \Pi$ . (irrational!)

Hence  $g$  is invertible and  $i_{C(\Pi)}(g)$  is invertible in  $A_0$ .

However

$$i_{C(\Pi)}(g) = \sum_{k=-n}^n i_{C(\Pi)}(\theta_k(h)) = \sum_{k=-n}^n u^k * i_{C(\Pi)}(h) * u^{-k} \in I.$$

Thus  $I = A_0$

If  $\theta$  is rational, there exist  $A_0$  is not simple.



In this proof, we defined  $\Phi: A_0 \rightarrow A_0$ . this is called conditional expectation.

Def (Conditional expectation)

$A: C^*$ -alg,  $B \subset A: C^*$ -subalgebra.

$\Phi: A \rightarrow B$  is conditional expectation.

if  $\Phi$  is positive linear map, and  $\Phi|_B = \text{id}_B$ .

In this case,  $A = A_0$ ,  $B = \text{im}(C(\pi))$ .

Conditional expectation is often used to show that.

$C^*$ -algebra is simple. When we analyse Cuntz algebra,

Conditional expectation is represented as a limit of.

inner endomorphisms. And using it, we can prove.

that Cuntz algebra is purely infinite, particularly simple.

Lastly, I'll introduce some property and example of  $A_0$ .

o Examples of  $A_0$ .

Ex. 1).  $\pi := \mathbb{R}/\mathbb{Z}$ ,  $U, V \in B(L^2(\pi))$ .

$$(U\xi)(t) := e^{2\pi i t} \xi(t), \quad (V\xi)(t) := \xi(t+\theta).$$

$$\text{then } VU = e^{2\pi i \theta} UV$$

Ex. 2).  $m, n$  are coprime.  $e_{ij}$  is matrix unit of  $M(n, \mathbb{C})$ .

$$U = \sum_{k=1}^n e^{2\pi i k(n-1)/n} e_{kk}, \quad V = \sum_{k=1}^n e_{k, k+1}$$

$$V = \sum_{k=1}^n e_{k, k+1}$$

$$\begin{pmatrix} e^{2\pi i} & & \\ & \ddots & \\ & & e^{2\pi i(n-1)/n} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 0 \end{pmatrix}$$



◦ Some properties of  $A_\theta$ .

Thm. (Powers, etc.)

irrational rotation algebra  $A_\theta$  is simple and not type I

Thm (Pimsner - Voiculescu, Rieffel).

$\theta_1, \theta_2$  is irrational s.t.  $0 < \theta_i < 1, (i=1,2)$  ..

1.  $A_{\theta_1} \simeq A_{\theta_2} \Leftrightarrow \theta_1 = \theta_2, \text{ or } \theta_1 + \theta_2 = 1$ .

2.  $A_{\theta_1} \hat{\otimes} K(\mathcal{H}) \simeq A_{\theta_2} \hat{\otimes} K(\mathcal{H})$

$$\Leftrightarrow \theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d}$$

Reference.

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