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Theorem

X: A unital Banach algebra.

If $a \in X$, then $r(a) = \inf_{n \geq 1} \|a^n\|^{\frac{1}{n}} = \lim_{n \to +\infty} \|a^n\|^{\frac{1}{n}}$.

pf [Murphy, Thm 1.2.7]

$a = 0 \Rightarrow \{|\lambda| \mid \lambda \in \sigma(a)\} = \{0\}$, $\{\|a^n\|^{\frac{1}{n}} \mid n \in \mathbb{N}\} = \{0\}$.

Hence, $r(a) = 0$, $\inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} = 0$, $\lim_{n \to +\infty} \|a^n\|^{\frac{1}{n}} = 0$

$$r(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} = \lim_{n \to +\infty} \|a^n\|^{\frac{1}{n}}.$$

Next, we concider $a \neq 0$.

We set $p(\lambda) = \lambda^n$, $p(a) = a^n$ $(a \in X)$.

Hence.

$$\sigma(a^n) = \{\lambda^n \mid \lambda \in \sigma(a)\}.$$

so, $\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n)$, $|\lambda^n| \leq \|a^n\|$ is true.

$|\lambda| \leq \|a^n\|^{\frac{1}{n}}$ and therefore, $r(a) \leq \inf_{n \geq 1} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n \to +\infty} \|a^n\|^{\frac{1}{n}}$.

We define the set $\Delta := \{z \in \mathbb{C} \mid |\lambda| < \frac{1}{r(a)}\}$. (If $r(a) = +\infty$, then, $\frac{1}{r(a)} = +\infty$. we promise)

If $\lambda \in \Delta$, $r(a) < \frac{1}{|\lambda|}$ and therefore $1 - \lambda a \in \mathrm{Inv}(X)$.

If $\tau \in X^*$, then the map $f : \Delta \longrightarrow \mathbb{C}$, $\lambda \longmapsto \tau((1-\lambda a)^{-1})$

is analytic, so f can have a Laurent expansion (uniquely) such that

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n \quad (\lambda \in \Delta,\ \{\lambda_n\} \subset \mathbb{C})$$

Meanwhile, if $|\lambda| < \frac{1}{\|a\|}$, then $\|\lambda a\| < 1$, so

$$(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n \quad (\text{Neumann Seriese})$$

and therefore

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^n \tau(a^n)$$

From the uniqueness of the Laurent expansion $\lambda_n = \tau(a^n)\ (n \in \mathbb{N})$ follows. $\sum_{n=0}^{\infty} \tau(a^n)\lambda^n < +\infty$, and therefore

$\lim_{n \to +\infty} \tau(a^n)\lambda^n = 0$. $\therefore$ $\{\tau(a^n)\lambda^n\}$ is a bounded sequence.

For each $\lambda \in \Delta$, $|\tau(\lambda^n a^n)| < +\infty$, $\sup_{n \in \mathbb{N}} |\tau(\lambda^n a^n)| < +\infty$.

Hence, $\tau \in X^*$ is arbitary, $\exists M > 0$, $\forall n \in \mathbb{N}$, $\|\lambda^n a^n\| \leq M$ is true

and therefore, $\|a^n\|^{\frac{1}{n}} \leq \frac{M^{\frac{1}{n}}}{|\lambda|}$ $(\lambda \neq 0)$.

Consequently,

$$\limsup_{n\to+\infty} \|a^n\|^{\frac{1}{n}} \leq \frac{1}{|\lambda|} \quad \text{and} \quad |\lambda| < \frac{1}{r(a)}.$$

If, $|\lambda| \to \frac{1}{r(a)}$, then $\limsup\limits_{n\to+\infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$.

Hence,

$$\limsup_{n\to+\infty} \|a^n\|^{\frac{1}{n}} \leq r(a) \leq \inf_{n\in\mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n\to+\infty} \|a^n\|^{\frac{1}{n}}$$

$$\leq \limsup_{n\to+\infty} \|a^n\|^{\frac{1}{n}}$$

$$r(a) = \inf_{n\in\mathbb{N}} \|a^n\|^{\frac{1}{n}} = \lim_{n\to+\infty} \|a^n\|^{\frac{1}{n}}$$