数理科学特論 I Report

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Problem (Extension of Fourier transform from $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$)

Schwartz space

$$\mathcal{F}[f](\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} f(x)\,dx, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

(1) $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is well-defined.

(2) $\mathcal{F}$ is invertible and inverse of $\mathcal{F}$ is

$$\mathcal{F}^*[f](x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} f(\xi)\,d\xi.$$

(3) $\forall f, g \in \mathcal{S}(\mathbb{R}^d)$, $\langle \mathcal{F}[f], \mathcal{F}[g]\rangle_{L^2} = \langle f, g\rangle_{L^2}$.

Consequently, $\mathcal{F}$ has a unique extension to $L^2(\mathbb{R}^d)$ since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. ($\because C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$) //

Proof (1) By the following properties of Fourier transf (i), (ii):

$\forall f \in L^1(\mathbb{R}^d)$, $\forall \alpha \in \mathbb{Z}^d_{\geq 0}$,

$$\begin{cases} \text{(i) if } x^\alpha f \in L^1(\mathbb{R}^d), \text{ then } \mathcal{F}[f] \in C^{|\alpha|}(\mathbb{R}^d) \text{ and} \\ \qquad i^{|\alpha|}\partial_\xi^\alpha \mathcal{F}[f](\xi) = \mathcal{F}[x^\alpha f](\xi), \\ \text{(ii) if } f \in C^{|\alpha|}(\mathbb{R}^d), \text{ then } \xi^\alpha \mathcal{F}[f](\xi) = \mathcal{F}[(-i)^{|\alpha|}\partial_x^\alpha f](\xi) \end{cases}$$

and definition of $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$,

$\forall \alpha, \beta \in \mathbb{Z}^d_{\geq 0}$, $\forall \xi \in \mathbb{R}^d$, $\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$|\xi^\alpha \partial_\xi^\beta \mathcal{F}[f](\xi)| = |\mathcal{F}[\partial_x^\alpha[x^\beta f]](\xi)| \leq \frac{\|\partial_x^\alpha[x^\beta f]\|_{L^1}}{(2\pi)^{\frac{d}{2}}} < \infty.$$

Hence, $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is well-defined. ┘

(2) Idea of following proof is "Oscillatory integral".

$\varphi_\varepsilon(\xi) := e^{-\varepsilon \cdot \frac{\|\xi\|^2}{2}} \in \mathcal{S}(\mathbb{R}^d)$ and $\lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = 1$, $|\varphi_\varepsilon(x)| \leq 1$.

$\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}^*[\mathcal{F}[f]](x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int e^{i\langle \xi, x\rangle} \mathcal{F}[f](\xi)\,d\xi$$

$$= \frac{1}{(2\pi)^d} \int e^{i\langle \xi, x\rangle} \int e^{-i\langle y, \xi\rangle} f(y)\,dy\,d\xi$$

$$= \frac{1}{(2\pi)^d} \iint e^{i\langle \xi, x-y\rangle} f(y)\, dy\, d\xi$$

DCT
$$= \frac{1}{(2\pi)^d} \lim_{\varepsilon \to 0} \iint e^{i\langle \xi, x-y\rangle} \varphi_\varepsilon(\xi) f(y)\, dy\, d\xi$$

Fubini
$$= \frac{1}{(2\pi)^d} \lim_{\varepsilon \to 0} \int f(y) \left(\int e^{i\langle \xi, x-y\rangle} \varphi_\varepsilon(\xi)\, d\xi \right) dy$$

$x-y=-\varepsilon\zeta$
$$= \frac{1}{(2\pi)^d} \lim_{\varepsilon \to 0} \int f(x+\varepsilon\zeta) \left(\int e^{-i\langle \xi, \varepsilon\zeta\rangle} \varphi_\varepsilon(\xi)\, d\xi \right) \varepsilon^d\, d\zeta$$

$\xi = \frac{y}{\varepsilon}$
$$= \frac{1}{(2\pi)^d} \lim_{\varepsilon \to 0} \int f(x+\varepsilon\zeta) \left(\int e^{-i\langle y, \zeta\rangle} e^{-\frac{\|y\|^2}{2}}\, dy \right) d\zeta$$

$\mathcal{F}[e^{-\frac{\|\cdot\|^2}{2}}](\xi) = e^{-\frac{\|\xi\|^2}{2}}$
$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \cdot \lim_{\varepsilon \to 0} \int f(x+\varepsilon\zeta)\, e^{-\frac{\|\zeta\|^2}{2}}\, d\zeta$$

DCT
$$= \frac{1}{(2\pi)^{\frac{d}{2}}} f(x) \cdot \int e^{-i\langle 0, \zeta\rangle} e^{-\frac{\|\zeta\|^2}{2}}\, d\zeta$$

$$= f(x) \cdot \mathcal{F}[e^{-\frac{\|\cdot\|^2}{2}}](0)$$

$$= f(x)$$

$\therefore\ \forall f \in \mathcal{S}(\mathbb{R}^d),\ \mathcal{F}^*[\mathcal{F}[f]] = f. \cdots (*)$

Also,

$\forall f \in \mathcal{S}(\mathbb{R}^d),\ \mathcal{F}[\mathcal{F}^*[f]](\xi) = \mathcal{F}[\mathcal{F}[f](-\cdot)](\xi)$

$= \mathcal{F}^*[\mathcal{F}[f](-\cdot)](-\xi) \underset{(*)}{=} f(-(-\xi)) = f(\xi).$

$\therefore\ \forall f \in \mathcal{S}(\mathbb{R}^d),\ \mathcal{F}[\mathcal{F}^*[f]] = f.$ ┘

(3) $\forall f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle \mathcal{F}[f], \mathcal{F}[g] \rangle_{L^2} = \int \mathcal{F}[f](\xi)\, \overline{\mathcal{F}[g](\xi)}\, d\xi$$

$$= \frac{1}{(2\pi)^d} \int \int f(x) e^{-i\langle \xi, x\rangle} dx \int \overline{g(y)} e^{i\langle \xi, y\rangle} dy\, d\xi$$

$$\overset{\boxed{\text{DCT}}}{=} \frac{1}{(2\pi)^d}\lim_{\varepsilon\to 0}\int \varphi_\varepsilon(\xi)\int f(x)e^{-i\langle\xi,x\rangle}dx\int\overline{g(y)}e^{i\langle\xi,y\rangle}dy\,d\xi$$

$$\overset{\boxed{\text{Fubini}}}{=} \frac{1}{(2\pi)^d}\lim_{\varepsilon\to 0}\iint f(x)\overline{g(y)}\left(\int\varphi_\varepsilon(\xi)e^{i\langle\xi,y-x\rangle}d\xi\right)dx\,dy$$

$$\overset{\boxed{y-x=-\varepsilon\eta}}{=} \frac{1}{(2\pi)^d}\lim_{\varepsilon\to 0}\iint f(y+\varepsilon\eta)\overline{g(y)}\left(\int\varphi_\varepsilon(\xi)e^{-i\langle\xi,\varepsilon\eta\rangle}d\xi\right)\varepsilon^d d\eta\,dy$$

$$\overset{\boxed{\xi=\frac{\xi}{\varepsilon}}}{=} \frac{1}{(2\pi)^d}\lim_{\varepsilon\to 0}\iint f(y+\varepsilon\eta)\overline{g(y)}\left(\int e^{-\frac{\|\xi\|^2}{2}}e^{-i\langle\xi,\eta\rangle}d\xi\right)d\eta\cdot dy$$

$$\overset{\boxed{\mathcal{F}[e^{-\frac{\|\cdot\|^2}{2}}](\xi)=e^{-\frac{\|\xi\|^2}{2}}}}{=} \frac{1}{(2\pi)^{\frac{d}{2}}}\lim_{\varepsilon\to 0}\iint f(y+\varepsilon\eta)\overline{g(y)}e^{-\frac{\|\eta\|^2}{2}}d\eta\,dy$$

$$\overset{\boxed{\text{DCT}}}{=} \frac{1}{(2\pi)^{\frac{d}{2}}}\iint f(y)\overline{g(y)}e^{-\frac{\|\eta\|^2}{2}}d\eta\,dy$$

$$= \int f(y)\overline{g(y)}dy\cdot\mathcal{F}[e^{-\frac{\|\cdot\|^2}{2}}](0)$$

$$= \int f(y)\overline{g(y)}dy$$

$$= \langle f,g\rangle_{L^2}. \quad \lrcorner \quad //$$

Hence, $\mathcal{F}:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ is a isometric isomorphism in L^2-norm.

So, by BLT theorem, $\mathcal{F}$ has a unique extension to $L^2(\mathbb{R}^d)$ and, by isometricity of $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{F}:L^2\to L^2$ is unitary operater.

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