数理科学特論 I　Report

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X : top sp., $S \subset X$: set. I define $Cl(S)$ as closure of S.

Problem (spectrum of multiplication)

If $\varphi \in C(\mathbb{R}^d)$, then $\sigma(\varphi(X)) = Cl(\varphi(\mathbb{R}^d))$.

Show this. //

Proof "$\subset$" : I show that $\mathbb{C} \setminus Cl(\varphi(\mathbb{R}^d)) \subset \mathbb{C} \setminus \sigma(\varphi(X))$.

$\forall \lambda \in \mathbb{C} \setminus Cl(\varphi(\mathbb{R}^d))$, $\exists \varepsilon > 0$ s.t. $\forall x \in \mathbb{R}^d$, $|\varphi(x) - \lambda| > \varepsilon$ since $\mathbb{C} \setminus Cl(\varphi(\mathbb{R}^d))$ is an open set in $\mathbb{C}$.

Hence, $(\varphi - \lambda I)^{-1} \in C(\mathbb{R}^d)$, in particular, bounded on $\mathbb{R}^d$:

$\forall x \in \mathbb{R}^d$, $|(\varphi(x) - \lambda)^{-1}| < \varepsilon$.

Therefore $(\varphi - \lambda \cdot 1)^{-1}(X) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a bounded linear operator, of course $dom((\varphi - \lambda \cdot 1)^{-1}(X)) = L^2(\mathbb{R}^d)$, and it is clear that $(\varphi - \lambda \cdot 1)^{-1}(X) = (\varphi(X) - \lambda \cdot id_{L^2})^{-1}$, LHS is multiplication with symbol $(\varphi - \lambda \cdot 1)^{-1}$, RHS is inverse of $\varphi(X) - \lambda id_{L^2}$.

So, $\lambda \in \mathbb{C} \setminus \sigma(\varphi(X))$, that is, $\mathbb{C} \setminus Cl(\varphi(\mathbb{R}^d)) \subset \mathbb{C} \setminus \sigma(\varphi(X))$.

"$\supset$" : I show that $\varphi(\mathbb{R}^d) \subset \sigma(\varphi(X))$.

$\forall \lambda \in \varphi(\mathbb{R}^d)$, $\exists x_0 \in \mathbb{R}^d$ s.t. $\lambda = \varphi(x_0)$.

Since φ is continuous, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |\varphi(x) - \lambda| < \varepsilon$.

If $\lambda \in \mathbb{C} \setminus \sigma(\varphi(X))$, by definition of the resolvent,

$\exists (\varphi(X) - \lambda \cdot id_{L^2})^{-1} \in B(L^2(\mathbb{R}^d))$.

Hence $\forall f \in dom(\varphi(X))$, $\|(\varphi(X) - \lambda \cdot id_{L^2})^{-1}\| \cdot \|(\varphi(X) - \lambda \cdot id_{L^2}) f\|_{L^2} \geq \|f\|_{L^2}$.

But I consider $f(x) = \mathbb{1}_S(x)$, $S := \{x \in \mathbb{R}^d \mid |x - x_0| < \delta\}$.

Then, $\|f\|_{L^2} \leq \|(\varphi(X) - \lambda \cdot id_{L^2})^{-1}\| \cdot \left(\int_S |\varphi(x) - \lambda|^2 dx\right)^{\frac{1}{2}}$

$\leq \|(\varphi(X) - \lambda \cdot id_{L^2})^{-1}\| \cdot \varepsilon \cdot \|f\|_{L^2}$.

So $1 \leq \|(\varphi(X) - \lambda \cdot id_{L^2})^{-1}\| \cdot \varepsilon$ ($\forall \varepsilon > 0$), this is contradiction.

Therefore $\lambda \in \sigma(\varphi(X))$, that is, $\varphi(\mathbb{R}^d) \subset \sigma(\varphi(X))$.

$\therefore Cl(\varphi(\mathbb{R}^d)) \subset \sigma(\varphi(X))$. //