数理科学特論 I Report

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Examples of various convergence in $\mathcal{H}$, $B(\mathcal{H})$.

(1) weak convergence $\not\Rightarrow$ strong convergence in $\mathcal{H}$:

- $\mathcal{H} := \ell^2(\mathbb{N}) := \{(c_n)_{n=0}^{\infty} \mid c_n \in \mathbb{C}, \sum |c_n|^2 < \infty\}$
- $S \in B(\ell^2(\mathbb{N}))$: unilateral shift:

$$S(c_0, c_1, c_2, \cdots) = (0, c_0, c_1, \cdots)$$

$({}^{\forall}0 \neq)(c_n)_n \in \ell^2(\mathbb{N})$, $c^{(0)} := (c_n)_{n=0}^{\infty}$, $c^{(n)} := S^n c^{(0)}$.

Then, (i) $c^{(n)}$ converges to 0 weakly.

(ii) $c^{(n)}$ does not converge to 0 strongly.

$\because$ (i) ${}^{\forall}c' = (c'_n)_{n=0}^{\infty} \in \ell^2(\mathbb{N})$,

$$|\langle c^{(n)}, c' \rangle| \leq \sum_{k=0} |c_k \cdot \overline{c'}_{n-1+k}| \leq \|c^{(0)}\| \cdot \left(\sum_{k=n-1}^{\infty} |c'_k|^2\right)^{\frac{1}{2}} \to 0$$

as $n \to \infty$.

$\boxed{\text{Cauchy-Schwartz}}$

$\therefore c^{(n)} \xrightarrow{w} 0$.

(ii) Since S is an isometry operator,

$$\|c^{(n)}\| = \|S^n c^{(0)}\| = \|c^{(0)}\| \not\to 0 \quad \text{as } n \to \infty.$$

$\therefore c^{(n)} \overset{s}{\not\to} 0$. $\lrcorner$

(2) weak convergence $\not\Rightarrow$ strong convergence in $B(\mathcal{H})$.

(1) implies that unilateral shift

- converges to 0 weakly,
- does not converge to 0 strongly. $\lrcorner$

(3) strong convergence $\not\Rightarrow$ uniform convergence :

- $\mathcal{H} := L^2(\mathbb{R}, m) := \{ f : \mathbb{R} \to \mathbb{C} : \text{m'ble} \mid \int_{\mathbb{R}} |f|^2 < \infty \}$
- $F_n(x) := \mathbb{1}_{[n,n+1]}(x) := \begin{cases} 1 & x \in [n, n+1] \\ 0 & x \notin [n, n+1] \end{cases} \in L^2, \ n \in \mathbb{N}.$

$$M_{F_n} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) : \text{multiplication op}$$
$$f \longmapsto F_n \cdot f$$

⇝

(i) M_{F_n} converges to 0 strongly.

(ii) M_{F_n} does not converge to 0 uniformly.

$\because$ (i) $\forall f \in L^2(\mathbb{R})$,

$$\|M_{F_n} f\|^2 = \int |F_n(x) f(x)|^2 dx = \int \mathbb{1}_{[n,n+1]}(x) |f(x)|^2 dx \xrightarrow{\boxed{\text{DCT}}} 0$$

Therefore, $M_{F_n} \xrightarrow{s} 0$.

(ii) Generally, for L^∞- multiplication M_φ, $\varphi \in L^\infty(\mathbb{R})$, it follows that $\|M_\varphi\| = \|\varphi\|_{L^\infty}$.

(We can show this by using L^1-L^∞ duality.)

Since it is clear that $F_n \in L^\infty(\mathbb{R})$ and $\|F_n\|_{L^\infty} = 1$,

$$\|M_{F_n}\| = \|F_n\|_{L^\infty} = 1.$$

So, $M_{F_n} \overset{u}{\not\rightarrow} 0$. ┘