数理科学特論I Report

322201059 Hyuya Ito

Problem If (L, R) is a double centralizer for a C^*-algebra, show that $\|L\| = \|R\|$. //

Proof

Review (L, R) : double centralizer for a C^*-algebra $\mathcal{C}$

$:\Leftrightarrow$ A pair of bdd linear maps on $\mathcal{C}$ such that $\forall A, B \in \mathcal{C}$,

$$L(AB) = L(A)B,\quad R(AB) = AR(B),\quad R(A)B = AL(B).$$

$\forall A \in \mathcal{C}$,

$$
\begin{aligned}
\|R(A)\|^2 &\overset{\|a^*\|=\|a\| \,\&\, C^*\text{-identity}}{=} \|R(A)R(A)^*\| \overset{R(A)B = AL(B)}{=} \|A\cdot L(R(A)^*)\| \\
&\overset{\|AB\| \le \|A\|\cdot\|B\|}{\le} \|A\|\cdot\|L(R(A)^*)\| \\
&\underset{\text{bddness of } L}{\le} \|A\|\cdot\|L\|\cdot\|R(A)^*\| \underset{\|a^*\| = \|a\|}{=} \|A\|\cdot\|L\|\cdot\|R(A)\|.
\end{aligned}
$$

- $R(A) \neq 0$ ($\Leftrightarrow \|R(A)\| \neq 0$) : $\|R(A)\| \le \|L\|\cdot\|A\|$.
- $R(A) = 0$: It's obvious that $\|R(A)\| \le \|L\|\cdot\|A\|$.

Hence, $\|R\| \le \|L\|$.

Samely,

$$
\begin{aligned}
\|L(A)\|^2 &= \|L(A)^*L(A)\| = \|R(L(A)^*)A\| \\
&\le \|R(L(A)^*)\|\cdot\|A\| \\
&\le \|R\|\cdot\|L(A)^*\|\cdot\|A\| \le \|R\|\cdot\|L(A)\|\cdot\|A\|.
\end{aligned}
$$

So $\|L\| \le \|R\|$.

Therefore we get "$\|L\| = \|R\|$". //