数理科学特論I Report

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Problem $\mathcal{H}$: $\mathbb{C}$-Hilbert space, A on $\mathcal{H}$: self-adjoint.
$\{E_\lambda\}_\lambda$: corresponding spectral family.
Show the following equality : $\operatorname{supp}\{E_\lambda\} = \sigma(A)$. //

Proof (i) $\operatorname{supp}\{E_\lambda\} \supset \sigma(A)$:
I'll show that "$\mathbb{R} \setminus \operatorname{supp}\{E_\lambda\} \subset \rho(A)$".
$\forall \lambda \in \mathbb{R} \setminus \operatorname{supp}\{E_\lambda\}$, by definition of support of $\{E_\lambda\}$,
$\exists \varepsilon > 0$ s.t. $E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon} = 0$, that is,
$E((\lambda-\varepsilon, \lambda+\varepsilon]) = 0$, E : spectral measure.
So, $\forall f \in \mathcal{H}$, $\langle E((\lambda-\varepsilon, \lambda+\varepsilon])f, f \rangle = 0$.
Hence, $\frac{1}{\lambda - id(\mu)} \in L^\infty(E_f)$, $E_f(\cdot) := \langle E(\cdot)f, f \rangle$, $\forall f \in \mathcal{H}$

Therefore, $(\lambda I - A) \cdot \int \frac{1}{\lambda - id(\mu)} dE(\mu)$

$$= \int (\lambda - id(\mu))\, dE(\mu) \cdot \int \frac{1}{\lambda - id(\mu)} dE(\mu)$$

$$= \int (\lambda - id(\mu)) \cdot \frac{1}{\lambda - id(\mu)} dE(\mu) = \int_{\mathbb{R}} 1\, dE(\mu) = I$$

Samely, $\int \frac{1}{\lambda - id(\mu)} dE(\mu) \cdot (\lambda I - A) = I$.

$\therefore \lambda \in \rho(A)$. ┘

$1 = \|(\alpha - A)^{-1}(\alpha - A) f_\varepsilon\|^2$
$\leq c \cdot \|(\alpha - A) f_\varepsilon\|^2 < \varepsilon^2$

(ii) $\operatorname{supp}\{E_\lambda\} \subset \sigma(A)$: To except trivial case, suppose $A \neq 0$.
$\forall \alpha \in \operatorname{supp}\{E_\lambda\}$, $\forall \varepsilon > 0$, $E((\alpha-\varepsilon, \alpha+\varepsilon]) \neq 0$.
$\Rightarrow \exists f_\varepsilon \in D_A$ s.t. $\|f_\varepsilon\| = 1$, $E((\alpha-\varepsilon, \alpha+\varepsilon]) f_\varepsilon = f_\varepsilon$.

$$\|(\alpha - A) f_\varepsilon\|^2 = \int_{\mathbb{R}} |\alpha - \mu|^2 dE_{f_\varepsilon}(\mu) = \int_{(\alpha-\varepsilon, \alpha+\varepsilon]} |\alpha - \mu|^2 dE_{f_\varepsilon}(\mu)$$

$$\leq \varepsilon^2 \cdot \|E((\alpha-\varepsilon, \alpha+\varepsilon]) f_\varepsilon\|^2 = \varepsilon^2 \cdot \|f_\varepsilon\|^2 = \varepsilon^2.$$

If $\alpha \in \rho(A)$, $\forall \varepsilon > 0$. $1 \leq \|(\alpha - A)^{-1}\| \cdot \|(\alpha - A) f_\varepsilon\| \leq \|(\alpha - A)^{-1}\| \cdot \varepsilon$. This is contradiction. //