数理科学特論I Report

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Problem $L^1(\mathbb{R}, m)$ : Lebesgue $L^1$- space.

If we define convolution $*$ :

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, dy , \quad x \in \mathbb{R} , \ f, g \in L^1(\mathbb{R}),$$

then $L^1(\mathbb{R}, m)$ is a non-unital Abelian Banach $*$- algebra.

Show this. //

Proof Claim 1 $*$ is well-defined, that is,

$\forall f, g \in L^1$, $f * g \in L^1$.

$$\because \ \|f*g\| = \int_{\mathbb{R}} |(f*g)(x)|\, dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)\, g(y)\, dy \right| dx$$

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)| \cdot |g(y)|\, dy\, dx \ \underset{\uparrow}{=} \ \|f\| \cdot \|g\| < \infty$$

- $\int$ is trans invariant
- Fubini theorem

$\lrcorner$

Claim 2 $*$ is Abelian product.

$\because \ \forall f, g \in L^1(\mathbb{R})$,

$$(f*g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy = \int f(z)\, g(x-z)\, dz = (g*f)(x). \ \lrcorner$$

Norm inequality $\|f*g\| \leq \|f\| \cdot \|g\|$ was showed in Claim 1.

So, we showed that $L^1(\mathbb{R})$ is an Abelian Banach $*$- algebra.

Next, we have to show that $L^1(\mathbb{R})$ does not have a unit.

First, we introduce "the Fourier transform".

$\forall f \in L^1(\mathbb{R})$, $\mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$ : Fourier trans of $f$.

( Rem $f(x)e^{-i\xi x} \in L^1$, so $\mathcal{F}$ is well-defined.)

Claim 3 (convolution theorem)

$\forall f, g \in L^1(\mathbb{R})$, $\mathcal{F}[f*g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$. //

$$
\begin{aligned}
\because \quad \mathcal{F}[f*g](\xi) &= \int_{\mathbb{R}} (f*g)(x) e^{-i\xi x} dx \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x-y) g(y) dy \right) e^{-i\xi x} dx \\
&\overset{\boxed{\text{Fubini thm}}}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(y) e^{-i\xi(x-y)} \cdot e^{-i\xi y} dy dx \\
&\overset{\text{trans-invariant}}{=} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x) e^{-i\xi x} dx \right) g(y) e^{-i\xi y} dy \\
&= \mathcal{F}[f](\xi) \cdot \mathcal{F}[g](\xi).
\end{aligned}
$$

Claim 4 (Riemann-Lebesgue)

$\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ i.e. $\mathcal{F}[f](x) \to 0$ $(|x| \to \infty)$. //

$\because \quad |\mathcal{F}[f](\xi + h) - \mathcal{F}[f](\xi)| \leq \int_{\mathbb{R}} |f(x)| |e^{-ihx} - 1| dx \xrightarrow{\boxed{\text{DCT}}} 0$

$\therefore$ $\mathcal{F}[f]$ is continuous.

decay : We can show it by using approximation of $L^1$-func by simple function.

$\forall f \in L^1(\mathbb{R})$, $\forall \varepsilon > 0$, $\exists s = \sum_{k=0}^{m} a_n A_n$ : simple func

s.t. $A_n$: interval, $\int |f - s| < \varepsilon$.

$\mathcal{F}[s](\xi) \to 0$ as $|\xi| \to \infty$ is easy.

By the way,

$$\left|\mathcal{F}[f](\xi) - \mathcal{F}[s](\xi)\right| \leq \int |f-s| < \varepsilon.$$

So, if we adjust $s$ such that

$s$ satisfies $\int |f-s| < \frac{\varepsilon}{2}$ and

$\exists R > 0$ s.t. $|\xi| \geq R$, $|\mathcal{F}[s](\xi)| < \frac{\varepsilon}{2}$

then, $|\xi| \geq R$,

$$|\mathcal{F}[f](\xi)| \leq |\mathcal{F}[f](\xi) - \mathcal{F}[s](\xi)| + |\mathcal{F}[s](\xi)| \leq \int |f-s| + |\mathcal{F}[s](\xi)| < \varepsilon.$$

In other word $\mathcal{F}[f](\xi) \to 0$ as $|\xi| \to \infty$. ┘

<u>Claim 5</u> $f(x) = e^{-\frac{x^2}{2}} \Rightarrow \mathcal{F}[f](\xi) = \sqrt{2\pi}\, e^{-\frac{\xi^2}{2}}$. //

$\because$ $\mathcal{F}[f](\xi) = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} e^{-i\xi x} dx = \int_{\mathbb{R}} e^{-\frac{1}{2}(x+i\xi)^2} dx \cdot e^{-\frac{\xi^2}{2}}$

we can show this claim by calculating a complex integral of $e^{-\frac{1}{2}(x+i\xi)^2}$. ┘

☆ <u>Goal</u> $L^1$ does not have an unit element for the convolution.

$\because$ If $L^1$ has it, then $\forall f \in L^1$, $\mathbb{1} * f = f$ $(\mathbb{1} \in L^1)$.

In particular, we consider $f(x) = e^{-\frac{x^2}{2}}$, and by <u>Claim 3, 5</u>,

$\mathcal{F}[\mathbb{1}] \cdot \mathcal{F}[f] = \mathcal{F}[f]$, $\mathcal{F}[f] > 0$.

Therefore, $\mathcal{F}[\mathbb{1}] \equiv 1$.

But, this is contradiction by <u>Claim 4</u>. ┘ //