数理科学特論 I Report

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Problem $\mathcal{H}$: $\mathbb{C}$-Hilbert sp, $(A, D(A))$: densely defined such that $\exists C \in \mathbb{R}$: $\forall f \in D(A)$, $\|Af\| \leq C\|A\|$.
Then $\exists$ natural continuous extension $\bar{A}$ of A s.t. $Q(D(A)) = \mathcal{H}$.
This extension satisfies $\bar{A} \in \mathcal{B}(\mathcal{H})$ s.t. $\|\bar{A}\| \leq C$.
Construct this natural extension and show that $\|\bar{A}\| \leq C$. //

Proof I will construct it by "strong convergence limit".
$\forall f \in \mathcal{H}$, since $D(A)$ is dense in $\mathcal{H}$,
$\exists \{f_n\} \subset \mathcal{H}$ s.t. $\|f_n - f\|_{\mathcal{H}} \to 0$ as $n \to \infty$.
By an assumption of bddness of A on $D(A)$,
so $\|Af_n - Af_m\| \leq C \cdot \|f_n - f_m\| \to 0$ as $n, m \to \infty$,
, that is, $\{Af_n\} \subset \mathcal{H}$ is Cauchy sequence in $\mathcal{H}$.
By completeness of $\mathcal{H}$, there exist $\lim_{n\to\infty} Af_n$ in norm.
Let be $\bar{A}$ operator defined by $\|\bar{A}f - Af_n\| \to 0$ as $n \to \infty$
such that $f \in \mathcal{H}$, $\{f_n\} \subset D(A)$: $\|f_n - f\| \to 0$ as $n \to \infty$.

<u>Claim 1</u> $\bar{A}$ is well-defined, that is, value of $\bar{A}$ does not depend on choice of convergence sequence.

$\because$ $\forall f \in \mathcal{H}$, $\exists \{g_n\}, \{h_n\} \subset D(A)$ s.t. $\|f - g_n\|, \|f - h_n\| \to 0$.

$G := \lim_{n\to\infty} Ag_n$, $H := \lim_{n\to\infty} Ah_n$,

$$\|G - H\| = \lim_{n\to\infty} \|Ag_n - Ah_n\|$$
$$\leq C \cdot \lim_{n\to\infty} \|g_n - h_n\| \leq C \cdot \lim_{n\to\infty} (\|g_n - f\| + \|f - h_n\|) = 0$$

Therefore, $\bar{A}$ is well-defined. ⌟

<u>Claim 2</u> $\|\bar{A}\| \leq C$

$\because$ $\forall f \in \mathcal{H}$, $\exists \{f_n\} \subset D(A)$ s.t. $\|f_n - f\| \to 0$.

$\therefore$ $\|\bar{A}f\| = \lim_{n\to\infty} \|Af_n\| \leq C \cdot \lim_{n\to\infty} \|f_n\| = C \cdot \|f\|$. ⌟ //