Exercise 3.3.3 In order to show $(C_0(\Omega), G, \theta)$ is a C^*-dynamical system, all we have to do is to check θ is a continuous homomorphism.

homomorphism : $[\theta_{xy} f](\xi) = f((xy)^{-1}\xi) = f(y^{-1}x^{-1}\cdot\xi) = f(y^{-1}\cdot x^{-1}\xi) = [\theta_y f](x^{-1}\xi)$

$= [\theta_x(\theta_y f)](\xi)$ $\quad (x, y \in G,\ f \in f_0(\Omega),\ \xi \in \xi)$

$\therefore\ \theta_{xy} f = \theta_x \theta_y f$

Thus, $\theta(xy) = \theta(x)\theta(y)$. This shows θ keeps product.

Hence, θ is a homomorphism.

Continuity : It suffices to show θ is continuous in $1 \in G$. We prove if net $\{g_\lambda\}_{\lambda \in \Lambda}$ converges to $1 \in G$, then $\| \theta_{g_\lambda}(f) - \theta_1(f) \|_{C_0(\Omega)} \to 0$ $\quad ({}^\forall f \in C_0(\Omega))$

If this failed, ${}^\exists f \in C_0(\Omega)$, ${}^\exists \varepsilon > 0$, s.t. ${}^\forall \lambda \in \Lambda$ ${}^\exists n_\lambda \geqq \lambda$, $\| \theta_{g_{n_\lambda}}(f) - f \|_{C_0(\Omega)} \geqq \varepsilon$

Hence, for any $\lambda \in \Lambda$, there exists $\xi_\lambda \in \Omega$ $| f(g_{n_\lambda}^{-1}\xi_\lambda) - f(\xi_\lambda) | \geqq \varepsilon$ $\cdots(*)$

Since f is in $C_0(\Omega)$, then $K = \{ \xi \in \Omega \mid |f(\xi)| \geqq \frac{\varepsilon}{2} \}$ is compact.

By $(*)$, either $g_{n_\lambda}^{-1}\xi_\lambda$ or ξ_λ is contained in K.

Since G is locally compact, there exists a compact neighborhood V of 1 and g_{n_λ} is in V for "large" enough $\lambda \in \Lambda$ by $g_{n_\lambda} \to 1$.

Eventually, $\xi_\lambda \in VK$ for "large" enough $\lambda \in \Lambda$.

Moreover, VK is a compact set. Thus, $\{\xi_\lambda\}$ has a subnet $\{\xi_{\varphi(\mu)}\}_{\mu \in M}$ which converges to $\xi_0 \in VK$

$\therefore\ g_{n_{\varphi(\mu)}}^{-1} \cdot \xi_{\varphi(\mu)} \to 1 \cdot \xi_0 = \xi_0$

This contradicts $(*)$.

Reference Crossed Products of C^*-Algebras (Dana P. Williams)