Exercise 2.1.6 • First of all, we prove that if $\|A\|<1$ $(A\in\mathcal{C})$, then $1-A$ is invertible in $\mathcal{C}$, with

$(1-A)^{-1}=\sum_{n=0}^{\infty}A^n$ … $(*)$

$(\because)$ We define $S_N=\sum_{n=0}^{N}A^n$.

By $\sum_{n=0}^{\infty}\|A^n\|\leqq\sum_{n=0}^{\infty}\|A\|^n<\infty$ $(\because \|A\|<1)$, there exists $S=\lim_{N\to\infty}S_N$.

$S_N(1-A)=(1-A)S_N=\sum_{n=0}^{N}(A^n-A^{n+1})=1-A^{N+1}\to 1$ $(N\to\infty)$ $(\because \|A\|<1)$

Thus, we get $S(1-A)=(1-A)S=1$

$\therefore$ $(1-A)$ is invertible and $(1-A)^{-1}=S=\sum_{n=0}^{\infty}A^n$

• We show $\mathrm{Inv}(\mathcal{C})$ is a open set in $\mathcal{C}$.

We take a arbitrary $A\in\mathrm{Inv}(\mathcal{C})$. If we show $\underbrace{B(A,\frac{1}{\|A^{-1}\|})}_{\text{open ball with center } A \text{ and radius } \frac{1}{\|A^{-1}\|}}\subset\mathrm{Inv}(A)$, then this statement will be done.

For any $T\in B(A,\frac{1}{\|A^{-1}\|})$, we get $\|1-A^{-1}T\|=\|A^{-1}(A-T)\|\leqq\|A^{-1}\|\,\|A-T\|<1$

Hence, $A^{-1}T$ is invertible by $(*)$.

$\therefore$ $T=A(A^{-1}T)$ is also invertible, so $T\in\mathrm{Inv}(\mathcal{C})$ ☒

• Lastly, we show the map $u:\mathrm{Inv}(\mathcal{C})\to\mathcal{C}$, $A\mapsto A^{-1}$ is differentiable.

For any $A\in\mathrm{Inv}(\mathcal{C})$, we define the map $f:\mathcal{C}\to\mathcal{C}$, $B\mapsto -A^{-1}BA^{-1}$.

Obviously, f is a bounded linear operator.

Moreover, $\lim_{C\to 0}\frac{\|(A+C)^{-1}-A^{-1}-f(C)\|}{\|C\|}=\lim_{C\to 0}\frac{\|(A+C)^{-1}-A^{-1}+A^{-1}CA^{-1}\|}{\|C\|}$

$=\lim_{C\to 0}\frac{\|(1+A^{-1}C)^{-1}A^{-1}-A^{-1}+A^{-1}CA^{-1}\|}{\|C\|}\leqq\lim_{C\to 0}\frac{\|(1+A^{-1}C)^{-1}-1+A^{-1}C\|\,\|A^{-1}\|}{\|C\|}$ … ①

Here, $\|(1+A^{-1}C)^{-1}-1+A^{-1}C\|\leqq\|\sum_{n=0}^{\infty}(-1)^n(A^{-1}C)^n-1+A^{-1}C\|$ $(\because (*))$

$=\|\sum_{n=2}^{\infty}(-1)^n(A^{-1}C)^n\|$

$\leqq\sum_{n=2}^{\infty}\|A^{-1}C\|^n=\frac{\|A^{-1}C\|^2}{1-\|A^{-1}C\|}$ $(\because \|C\|$ is small enough$)$

$\leqq 2\|A^{-1}C\|^2$ $(\because \|C\|$ is small enough$)$

Therefore, $① \leqq \lim_{C\to 0} \frac{2\|A^{-1}C\|^2\|A^{-1}\|}{\|C\|} \leqq \lim_{C\to 0} \frac{2\|A^{-1}\|^3\|C\|^2}{\|C\|} = \lim_{C\to 0} 2\|A^{-1}\|^3\|C\| = 0$

$\therefore$ We get $\lim_{C\to 0} \frac{\|(A+C)^{-1} - A^{-1} - f(C)\|}{\|C\|} = 0$

This indicates $u: \mathrm{Inv}\,\mathcal{C} \to \mathcal{C}$, $A \longmapsto A^{-1}$ is differentiable and $u'(A) = f$ ▥