Banach-Alaoglu Theorem $\mathcal{B}$ (unit ball of $\mathcal{C}^*$) is w^*-compact.

pf. Let B be the unit ball of $\mathcal{C}$.

For each $f \in B$, we define $\mathbb{C}_1^f = \{z \in \mathbb{C} \mid |z| \leq 1\}$.

$\prod_{f\in B} \mathbb{C}_1^f$ (endowed with the product topology) is compact by Tychonoff's theorem, since $\mathbb{C}_1^f$ is a compact set.

We define $\Lambda : \mathcal{B} \longrightarrow \prod_{f\in B} \mathbb{C}_1^f$.

$$\varphi \longmapsto \varphi|_B$$

Let's check well-definedness: For all $f \in B$, $\|\varphi(f)\| \leq \|\varphi\|\|f\| \leq 1$ ($\because$ $\varphi \in \mathcal{B}$, $f \in B$)

Thus, $\varphi(f)$ is in $\mathbb{C}_1^f$. $\therefore$ $\varphi|_B \in \prod_{f\in B} \mathbb{C}_1^f$

We show Λ is one-to-one: We assume $\Lambda(\varphi_1) = \Lambda(\varphi_2)$. By definition $\varphi_1(f) = \varphi_2(f)$ ($\forall f \in B$)

For all $f \in \mathcal{C}$, $f \neq 0$, $\frac{f}{\|f\|}$ is in B, so $\varphi_1\left(\frac{f}{\|f\|}\right) = \varphi_2\left(\frac{f}{\|f\|}\right)$. This implies $\varphi_1(f) = \varphi_2(f)$

$\therefore$ $\varphi_1(f) = \varphi_2(f)$ ($\forall f \in \mathcal{C}$) Thus, $\varphi_1 = \varphi_2$

Moreover, a net $\{\varphi_\alpha\}_{\alpha\in A}$ in $\mathcal{B}$ converges in w^*-topology to φ in $\mathcal{B}$

$$\Longleftrightarrow \lim_{\alpha\in A} \varphi_\alpha(f) = \varphi(f) \quad (\forall f \in \mathcal{C}) \quad \Longleftrightarrow \lim_{\alpha\in A} \Lambda(\varphi_\alpha)(f) = \Lambda(\varphi)(f) \quad (\forall f \in B)$$

$$\Longleftrightarrow \lim_{\alpha\in A} \Lambda(\varphi_\alpha) = \Lambda(\varphi) \quad \text{in } \prod_{f\in B} \mathbb{C}_1^f$$

This implies that Λ is a homeomorphism between $\mathcal{B}$ and $\Lambda(\mathcal{B})$.

If we prove $\Lambda(\mathcal{B})$ is closed, then $\Lambda(\mathcal{B})$ is compact since $\prod_{f\in B} \mathbb{C}_1^f$ is compact and we get $\mathcal{B}$ is compact. Thus, we will show $\Lambda(\mathcal{B})$ is closed.

We assume that a net $\{\Lambda(\varphi_\alpha)\}_{\alpha\in A}$ in $\Lambda(\mathcal{B})$ converges to ψ in $\prod_{f\in B} \mathbb{C}_1^f$.

We define $\hat{\psi} : \mathcal{C} \to \mathbb{C}$

$$f \longmapsto \|f\|\psi\left(\frac{f}{\|f\|}\right)$$

We show $\hat{\psi} \in \mathcal{B}$. For any $f, g \in B$, $a, b \in \mathbb{C}$,

$$\psi(af+bg) = \lim_{\alpha\in A} \Lambda(\varphi_\alpha)(af+bg) \quad (\because \{\Lambda(\varphi_\alpha)\}_{\alpha\in A} \text{ converges to } \psi \text{ in } \prod_{f\in B} \mathbb{C}_1^f)$$

$$= a \lim_{\alpha\in A} \Lambda(\varphi_\alpha)(f) + b \lim_{\alpha\in A} \Lambda(\varphi_\alpha)(g)$$

$$= a\psi(f) + b\psi(g)$$

Thus, by the form of $\hat{\varphi}$, $\hat{\varphi}$ is linear map.

Moreover, For $f \in \mathcal{C}$, $\|\hat{\varphi}(f)\| = \|f\| \, |\varphi(\frac{f}{\|f\|})| \leq \|f\|$ $\quad (\because) \ \varphi(\frac{f}{\|f\|}) \in \mathbb{C}_1^f$

$\therefore \hat{\varphi} \in \mathcal{B}$.

For all $f \in B$, $\Lambda(\hat{\varphi})(f) = \hat{\varphi}(f) = \|f\| \varphi\left(\frac{f}{\|f\|}\right) = \varphi(f)$ $\quad (\because f \in B)$

Therefore, we get $\Lambda(\hat{\varphi}) = \varphi$ in $\prod_{f \in B} \mathbb{C}_1^f$

$\therefore \varphi \in \Lambda(\mathcal{B})$

Thus, $\Lambda(\mathcal{B})$ is closed.

Reference · Banach Algebra Techniques in Operator Theory. (Ronald G. Douglas)