No. 1

DATE

062101776 Hinata Yoshida

Exercise 4.4

If $u_0, u_1, \cdots$ has generating function $U(s)$ and $v_0, v_1, \cdots$ has generating function $V(s)$, find $V(s)$ in terms of $U(s)$ when (a) $v_n = 2u_n$, (b) $v_n = u_n + 1$, (c) $v_n = n u_n$.

(a)
$$\begin{aligned} V(s) &= v_0 + v_1 s + v_2 s^2 + \cdots \\ &= 2(u_0 + u_1 s + u_2 s^2 + \cdots) \\ &= \underline{2U(s)} \; /\!/ \end{aligned}$$

(b)
$$\begin{aligned} V(s) &= v_0 + v_1 s + v_2 s^2 + \cdots \\ &= (u_0 + u_1 s + u_2 s^2 + \cdots) + (1 + s + s^2 + \cdots) \\ &= U(s) + \underline{(1-s)^{-1}} \; /\!/ \quad \text{if } |s| < 1. \end{aligned}$$

(c)
$$\begin{aligned} V(s) &= v_0 + v_1 s + v_2 s^2 + \cdots \\ &= u_1 s + 2u_2 s^2 + 3u_3 s^3 + \cdots \\ &= \sum_{n=0}^{\infty} n u_n s^n \\ &= s \sum_{n=1}^{\infty} n u_n s^{n-1} \\ &= s \sum_{n=1}^{\infty} \frac{d}{ds}(u_n s^n) \\ &= s \times \frac{d}{ds}\left(\sum_{n=1}^{\infty} u_n s^n\right) \\ &= s \times \frac{d}{ds}\left(\sum_{n=0}^{\infty} u_n s^n\right) \\ &= \underline{s U'(s)} \; /\!/ \end{aligned}$$

Exercise 4.31

If X has the negative binomial distribution with parameters n and p, show that

$$\mathbb{E}(X) = n/p, \qquad \text{var}(X) = nq/p^2$$

where $q = 1-p$.

By (4.17),

$$G_X(s) = \left(\frac{ps}{1-qs}\right)^n \qquad \text{if } |s| < q^{-1}.$$

(i) When $n \geq 2$,

$$G_X'(s) = \frac{d}{ds}\left(\frac{ps}{1-qs}\right)^n$$

$$= n\left(\frac{ps}{1-qs}\right)^{n-1} \times \left(\frac{ps}{1-qs}\right)'$$

$$= n\left(\frac{ps}{1-qs}\right)^{n-1} \times \left\{p(1-qs)^{-1} + pqs(1-qs)^{-2}\right\}$$

$$G_X''(s) = \frac{d}{ds}\left[n\left(\frac{ps}{1-qs}\right)^{n-1} \times \left\{p(1-qs)^{-1} + pqs(1-qs)^{-2}\right\}\right]$$

$$= n(n-1)\left(\frac{ps}{1-qs}\right)^{n-2}\left(\frac{ps}{1-qs}\right)' \times \left\{p(1-qs)^{-1} + pqs(1-qs)^{-2}\right\}$$

$$+ n\left(\frac{ps}{1-qs}\right)^{n-1} \times \left\{p(1-qs)^{-1} + pqs(1-qs)^{-2}\right\}'$$

$$= n(n-1)\left(\frac{ps}{1-qs}\right)^{n-2}\left\{p(1-qs)^{-1} + pqs(1-qs)^{-2}\right\}^2$$

$$+ n\left(\frac{ps}{1-qs}\right)^{n-1}\left\{2pq(1-qs)^{-2} + 2pq^2s(1-qs)^{-3}\right\}$$

According to the Theorem 4.23, we have that

$$\mathbb{E}(X) = G_X'(1)$$

$$= n\left(\frac{p}{1-q}\right)^{n-1} \times \left\{p(1-q)^{-1} + pq(1-q)^{-2}\right\}$$

$$= n \times \left(1 + \frac{q}{p}\right) = \frac{n}{p}$$

According to (4.22) and the Theorem 4.23, we have that

$$\mathrm{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= G_X''(1) + G_X'(1) - \left\{G_X'(1)\right\}^2$$

$$= n(n-1)\left(\frac{p}{1-q}\right)^{n-2}\left\{p(1-q)^{-1} + pq(1-q)^{-2}\right\}^2$$

$$+ n\left(\frac{p}{1-q}\right)^{n-1}\left\{2pq(1-q)^{-2} + 2pq^2(1-q)^{-3}\right\}$$

$$+ G_X'(1) - \left\{G_X'(1)\right\}^2$$

$$= n(n-1)\left(1+\frac{q}{p}\right)^2 + n\left(\frac{2q}{p}+\frac{2q^2}{p^2}\right)$$

$$+ G_X'(1) - \{G_X'(1)\}^2$$

$$= \left(1+\frac{2q}{p}+\frac{q^2}{p^2}\right)n^2 + \left(-1-\frac{2q}{p}-\frac{q^2}{p^2}+\frac{2q}{p}+\frac{2q^2}{p^2}\right)n$$

$$+ G_X'(1) - \{G_X'(1)\}^2$$

$$= \left(1+\frac{2q}{p}+\frac{q^2}{p^2}\right)n^2 + \left(-1+\frac{q^2}{p^2}\right)n$$

$$+ G_X'(1) - \{G_X'(1)\}^2$$

$$= \left\{1+\frac{2(1-p)}{p}+\frac{(1-p)^2}{p^2}\right\}n^2 + \left\{-1+\frac{(1-p)^2}{p^2}\right\}n$$

$$+ G_X'(1) - \{G_X'(1)\}^2 \qquad (\because q = 1-p)$$

$$= \frac{n^2+(1-2p)n}{p^2} + \frac{n}{p} - \left(\frac{n}{p}\right)^2 \qquad \left(\because E(X) = \frac{n}{p}\right)$$

$$= \frac{n^2+(1-2p)n+np-n^2}{p^2}$$

$$= \frac{(1-p)n}{p^2} = \frac{nq}{p^2}$$

(ii) When $n=1$.

$$G_X(s) = \frac{ps}{1-qs}$$

$$G_X'(s) = p(1-qs)^{-1} + pqs(1-qs)^{-2} = \frac{p}{(1-qs)^2}$$

$$G_X''(s) = \left(p(1-qs)^{-2}\right)' = p(1-qs)^{-3} \times 2q = \frac{2pq}{(1-qs)^3}$$

According to the Theorem 4.23, we have that

$$\mathbb{E}(X) = G_X'(1)$$

$$= \frac{p}{(1-q)^2} = \frac{1}{p} = \frac{n}{p}$$

According to (4.22) and the Theorem 4.23, we have that

$$\mathrm{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= G_X''(1) + G_X'(1) - \{G_X'(1)\}^2$$

$$= \frac{2pq}{(1-q)^3} + \frac{p}{(1-q)^2} - \frac{p^2}{(1-q)^4}$$

$$= \frac{2q+p-1}{p^2}$$

$$= \frac{2q+1-q-1}{p^2} = \frac{1q}{p^2} = \frac{nq}{p^2}$$

(iii) when $n=0$.

$$G_X(s) = \left(\frac{ps}{1-qs}\right)^0 = 1.$$

$$G_X'(s) = 0, \quad G_X''(s) = 0$$

According to the Theorem 4.23, we have that

$$\mathbb{E}(X) = G_X'(1) = 0 = n/p$$

According to (4.22) and the Theorem 4.23, we have that

$$\begin{aligned} var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= G_X''(1) + G_X'(1) - \{G_X'(1)\}^2 \\ &= 0 + 0 - 0 \\ &= 0 = nq/p^2 \end{aligned}$$

By (i)~(iii), we can show that $\mathbb{E}(X) = n/p$, $var(X) = nq/p^2$