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Exercise 5.11

Let X be a random variable taking integer values such that $\mathbb{P}(X=k)=p_k$ for $k=\cdots,-1,0,1,\cdots$. Show that the distribution function of X satisfies

$$F_X(b)-F_X(a)=p_{a+1}+p_{a+2}+\cdots+p_b$$

for all integers a,b with $a<b$.

According to (5.10), for $a<b$,

$$\begin{aligned} F_X(b)-F_X(a) &= \mathbb{P}(a<X\leq b) \\ &= \mathbb{P}(a+1\leq X\leq b) \quad (\because a\in\mathbb{Z}) \\ &= p_{a+1}+p_{a+2}+\cdots+p_b \quad \square \end{aligned}$$

Exercise 5.46
Show that the gamma function $\Gamma(w)$ satisfies $\Gamma(w) = (w-1)\Gamma(w-1)$ for $w > 1$, and deduce that $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \cdots$.

$$\begin{aligned}\Gamma(w) &= \int_0^\infty x^{w-1} e^{-x} dx \\ &= \left[-x^{w-1} e^{-x}\right]_0^\infty + \int_0^\infty e^{-x} (w-1) x^{w-2} dx \\ &= (w-1) \int_0^\infty e^{-x} x^{w-2} dx \\ &= (w-1)\Gamma(w-1) \quad \square\end{aligned}$$

In a similar way,

$$\begin{aligned}\Gamma(n) &= (n-1)(n-2)\Gamma(n-2) \\ &= (n-1)(n-2)(n-3)\Gamma(n-3) \\ &\vdots\end{aligned}$$

$$\begin{aligned}\therefore \Gamma(n) &= (n-1)(n-2)\cdots 2 \cdot \Gamma(2) \\ &= (n-1)(n-2)\cdots 2 \cdot 1 \cdot \Gamma(1) \\ &= (n-1)! \cdot \Gamma(1)\end{aligned}$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$,

we can deduce that $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \cdots$.