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Solution to the Exercise 6.35 at page 90, [GW]

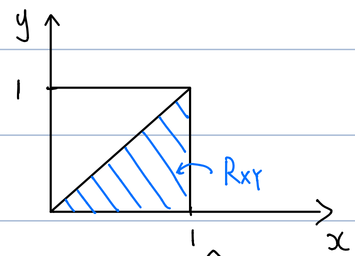
Exercise 6.35:

Let X and Y have joint density function

$$f(x,y) = \begin{cases} cx & \text{if } 0<y<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of the constant c and the marginal density function of X and Y. Are X and Y independent?

In order to find the value of c, one needs to draw the integration region R_{XY} first:



R_{XY} in the x-y plane

Recall that all joint density functions integrate to 1, so that

$$1 = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\right]dy = \int_0^1\left[\int_0^x cx\,dy\right]dx$$

$$= \int_0^1 (cx)(x)\,dx = \int_0^1 cx^2\,dx = c\left[\frac{x^3}{3}\right]_0^1 = \frac{c}{3}$$

One can easily conclude the value of the constant c is $\boxed{3}$.

Marginal density function of X, $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$

$$= \int_0^x 3x\,dy$$

$$= \boxed{3x^2,\ \text{for } x \in (0,1)}.$$

Thus, one has

$$f_X(x) = \begin{cases} 3x^2 & 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, one can obtain marginal density function of Y, $f_Y(y)$:

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \\
&= \int_y^1 3x\,dx \\
&= \frac{3}{2}x^2 \Big|_{x=y}^{x=1} \\
&= \boxed{\frac{3}{2}(1-y^2)\ ,\ \text{for } y \in (0,1)}.
\end{aligned}
$$

Thus, one has

$$
f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2) & 0<y<1 \\ 0 & \text{otherwise}. \end{cases}
$$

Since $3x \neq (3x^2)\frac{3}{2}(1-y^2)$ implying that $f_{XY}(x,y) \neq f_X(x)\, f_Y(y)$,

One can easily conclude that X and Y are $\boxed{\text{not independent}}$.