Variance of Discrete R.V and (a.) Continuous R.V

Date 2/6 (?/c)

(Prof's Note, P20)

Recall that $Var(X) := \mathbb{E}((X-\overbrace{\mathbb{E}(X)}^{\mu})^2) = \underbrace{\mathbb{E}(X^2) - \mathbb{E}(X)^2}$

in Discrete random variables.

Small proof

This is the useful formula we commonly see in texts.

Proof: Note that $\mathbb{E}(X)$ is just a number, which is mean, μ. We can simply solve this with algebra!

$$\mathbb{E}((X-\mu)^2) = \mathbb{E}(X^2 - 2\mu X + \mu^2)$$
$$= \mathbb{E}(X^2) - 2\mu \underbrace{\mathbb{E}(X)}_{\mu} + \mu^2$$
$$= \mathbb{E}(X^2) - 2\mu^2 + \mu^2$$
$$= \mathbb{E}(X^2) - \mu^2$$
$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \quad \square$$

	X	$-\mu$
X	X^2	$-X\mu$
$-\mu$	$-X\mu$	μ^2

$\Rightarrow X^2 - 2\mu X + \mu^2$

Exercise 2.3.8 [GW, P33]

Show that $\boxed{Var(aX+b) = a^2 Var(X)}$ for $a, b \in \mathbb{R}$

↳ Also in 6/23 class on VII: Moments

Proof: As $\mu = \mathbb{E}(X)$. Then, $\mathbb{E}(aX+b) = a\mu + b$.

$$Var(aX+b) = \mathbb{E}((aX+b-(a\mu+b))^2)$$
$$= \mathbb{E}((aX-a\mu)^2)$$
$$= \mathbb{E}(a^2(X-\mu)^2)$$
$$= a^2\,\mathbb{E}((X-\mu)^2)$$
$$= a^2\,Var(X) \quad \square$$

[Prof's notes, P46]

In (absolutely) Continuous random variables case, first let's define the mean, $\mathbb{E}(X)$. We can simply replace the sum, $\sum_{x \in X(\Omega)}^{n}$ with an Integral and Probability mass function (PMF), $P_X(x)$ with probability density function (PDF), $f_X(x)$.

$$\boxed{\mathbb{E}(X) := \underbrace{\int_{-\infty}^{\infty}}_{\text{Integral}} (x) \underbrace{f_X(x)}_{\text{PDF}}\, dx, \text{ whenever it converges absolutely.}}$$

Likewise, if X has PDF, $f_X(x)$ and $F_X(x)$, CDF (Cumulative distribution function)

$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x^2) f_X(x)\, dx - \int_{-\infty}^{\infty} (2\mu x) f_X(x)\, dx + \int_{-\infty}^{\infty} (\mu^2) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x^2) f_X(x)\, dx - 2\mu \underbrace{\int_{-\infty}^{\infty} x f_X(x)\, dx}_{\mathbb{E}(X)} + \mu^2 \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{1}$$

Shift the constant　　Shift the constant

$$= \int_{-\infty}^{\infty} (x^2) f_X(x)\, dx - 2\mu \cdot \mu + \mu^2 \cdot 1$$

$$= \int_{-\infty}^{\infty} (x^2) f_X(x)\, dx - \mu^2$$

$$= \underbrace{\int_{-\infty}^{\infty} (x^2) f_X(x)\, dx}_{\mathbb{E}(X^2)} - \underbrace{\left(\int_{-\infty}^{\infty} (x) f_X(x)\, dx\right)^2}_{\mu^2 = \mathbb{E}(X)^2}$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \quad \square$$

As you can see, many concepts in discrete valued r.v (Chapter 2 & 3) can be applied again in (a.) continuous r.v (Chapter 5 & 6) So highly recommend to study the early chapters, especially for discrete cases. ☺

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