Chapter 3 (and a bit from Chapter 2) 5/30(日)

(P24) of Prof's Notes / (P40) of [GW] (Linearity of Expectation)

Check! If X and Y are discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $a, b \in \mathbb{R}$, then

$$\mathbb{E}(aX + bY) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$$

Proof:

First, let's break this into 3 parts. We'll show...

① $\mathbb{E}(aX) = a\,\mathbb{E}(X)$

② $\mathbb{E}(X+b) = \mathbb{E}(X) + b$ ← useful properties in C2, prof's notes (P20)

③ $\mathbb{E}(aX+bY) = a\mathbb{E}(X) + b\,\mathbb{E}(Y)$

For ① aX and ② $X+b$, since they are functions of X, we can apply Law of subconscious statistican.

Recall: For $g: \mathbb{R} \to \mathbb{R}$, one has

$$\mathbb{E}(g(X)) = \sum_{x \in X(\Omega)} g(x) \underbrace{P_X(x)}_{P(X=x)}$$

whenever $\sum_{x \in X(\Omega)} |g(x)|\, P_X(x) < \infty$ (this sum converges absolutely.)

① $\mathbb{E}(aX) = a\,\mathbb{E}(X)$

$$\mathbb{E}(aX) = \sum_{x \in X(\Omega)} (ax) P_X(x) \quad \text{(Law of subconscious statistican)}$$

$$= a \underbrace{\sum_{x} x\, P_X(x)}_{\text{Definition of } \mathbb{E}(X) \equiv \text{MEAN} \equiv \text{EXPECTATION}}$$

$$= a\,\mathbb{E}(X)$$ //

② $\mathbb{E}(X+b) = \mathbb{E}(X) + b$

$$\mathbb{E}(X+b) = \sum_{x \in X(\Omega)} (x+b) P_X(x) \quad \text{(law of subconscious statistican)}$$

$$= \sum_{x} x\, P_X(x) + \sum_{x} b\, P_X(x)$$

$$= \sum_{x} x\, P_X(x) + b \underbrace{\sum_{x} P_X(x)}_{=\mathbb{P}(\Omega) = 1}$$ (See (pmf) on Prof's note, (P13))

$$= \mathbb{E}(X) + b$$ //

③ $\mathbb{E}(aX+bY) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$

$$\mathbb{E}(aX+bY) = \sum_{x\in X(\Omega)} \sum_{y\in Y(\Omega)} (ax+by)\, \underbrace{P_{X,Y}(x,y)}_{P(X=x,\,Y=y)}$$

$$= a\sum_x \sum_y x\, P_{X,Y}(x,y) + b \sum_x \sum_y P_{X,Y}(x,y)$$

$$= a \sum_x x\, P_X(x) + b \sum_y y\, P_Y(y)$$

$P_X(x) = \sum_y P_{X,Y}(x,y)$ and $P_Y(y) = \sum_x P_{X,Y}(x,y)$

$$= a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$$

$\square$

Remark: X and Y do not need to be Independent as $\mathbb{E}$ will act linearly on the set of discrete random variables.