Exercise relates 1 absolutely continuous random variables

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June 6, 2021

Question

Let X be an absolute continuous random variable, with continuous probability density f_X satisfying $f_X(x) = 0, \forall x \leq 0$.

We shall prove that $\mathbb{E}(X) = \int_0^\infty (1 - F_X(x))\,dx$, where F_X denotes the cumulative distribution function of X.

Proof 1

Recall

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^\infty x f_X(x)\,dx\,. \tag{1}$$

We calculate the L.H.S by the integration by parts,

$$\begin{aligned}\int_0^\infty x f_X(x)\,dx &= \int_0^\infty x(\frac{\mathrm{d}}{\mathrm{d}x} F_X(x))\,dx \\ &= \lim_{x_0\to\infty} [x F_X(x)|_0^{x_0} - \int_0^{x_0} F_X(x)\,dx\,].\end{aligned}$$

Here we know that $F_X(0)$ is finite.
Then,

$$xF_X(x)|_0^{x_0} - \int_0^{x_0} F_X(x)\,dx\, = x_0 F(x_0) - \int_0^{x_0} F(x)dx = x_0 - \int_0^{x_0} F(x)dx + x_0(F_X(x_0) - 1)$$

where the last term $x_0(F_X(x_0) - 1)$ is the remainder term. So we need to consider two situations, one that the remainder converges to 0 when $x \to \infty$, and one that it doesn't converge to 0.

First, suppose that $\lim_{y\to\infty}[y(F_X(y) - 1)] = 0$ does not hold.
Then,

$$|y(F_X(y) - 1)| = y(1 - F_X(y)) = y \int_y^\infty f_X(x)\,dx \leq \int_y^\infty x f_X(x)\,dx$$

$$\Longrightarrow \lim_{y\to\infty} \int_y^\infty x f_X(x)\,dx > 0 \Rightarrow \mathbb{E}(X) = \infty$$

Then we have

$$
\begin{aligned}
\int_0^\infty (1 - F_X(x))\, dx &= \int_0^\infty \int_x^\infty f_X(s)\, ds\, dx. \\
&= \lim_{y\to\infty} \int_0^y \int_x^\infty f_X(s)\, ds\, dx \\
&\geq \lim_{y\to\infty} \int_0^y \int_x^y f_X(s)\, ds\, dx \\
&= \lim_{y\to\infty} \int_0^y (\int_0^s dx) f_X(s)\, ds \\
&= \lim_{y\to\infty} \int_0^y s f_X(s)\, ds = \mathbb{E}(X).
\end{aligned}
$$

$$
\implies \int_0^\infty (1 - F_X(x))\, dx = +\infty = \mathbb{E}(X),
$$

showing the claim when $y(D_X(y) - 1)$ does not converge to 0 as $y \to \infty$.

Now, suppose that $\lim_{y\to\infty}[y(F_X(y) - 1)] = 0$, does hold. Then,

$$
\lim_{x_0\to\infty} (x_0 F(x_0) - \int_0^{x_0} F(x)dx) = \lim_{x_0\to\infty} (x_0 - \int_0^{x_0} F(x)dx)\,.
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}(X) &= \lim_{x_0\to\infty} [x_0 - \int_0^{x_0} F_X(x)\, dx] \\
&= \int_0^\infty (1 - F_X(x))\, dx,
\end{aligned}
$$

showing the claim when $\lim_{y\to\infty} y(F_X(y) - 1) = 0$. □

Proof 2

Show that for the integral,

$$
\begin{aligned}
&\int_0^\infty (1 - F_X(x))dx \\
:=& \int_0^\infty (1 - \int_0^x f_X(y)dy)dx \\
=& \int_0^\infty (\int_x^\infty f_X(y)dy)dx
\end{aligned}
$$

Then change the order of integration,

$$
\begin{aligned}
\implies& \int_0^\infty (\int_0^y f_X(y)dx)dy \\
=& \int_0^\infty (f_X(y) \int_0^y 1\ dx)dy \\
=& \int_0^\infty y f_X(y)dy \\
=& \mathbb{E}(X)\,.
\end{aligned}
$$

Therefore, we've proved the statement. □