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Date R3·7·5 No. 1/1

Binomial-Poisson limit

(Related to Ex 7.95, Ex 7.97b and Pb 8.7)

Let Z_n have the binomial distribution with parameters n and λ/n, where λ is fixed. Show that Z_n converges in distribution to the Poisson distribution, parameter λ, as $n \to \infty$.

Let $B(n,p)$ be a binomial distribution with parameters n and p.

Then, $\phi_{B(n,p)}(t) = \sum_{k=0}^{n} e^{itk} \binom{n}{k} p^k (1-p)^{n-k}$

Set $q = 1-p$.

$$\phi_{B(n,p)}(t) = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (e^{it}p)^k$$

$$= (q + pe^{it})^n$$

Secondly, let $P(\lambda)$ be a poisson distribution with parameter λ.

Then, $\phi_{P(\lambda)}(t) = \sum_{k=0}^{\infty} e^{itk} \frac{1}{k!} \lambda^k e^{-\lambda}$

$$= e^{-\lambda} \sum_{k=0}^{\infty} e^{itk} \frac{1}{k!} \lambda^k$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!}$$

$$= e^{-\lambda} \exp(\lambda e^{it})$$

$$= \exp(\lambda e^{it} - \lambda)$$

Consider $B(n, \frac{\lambda}{n})$,

$$\phi_{B(n,\frac{\lambda}{n})}(t) = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^{it}\right)^n$$

$$= \left(1 + \frac{\lambda e^{it} - \lambda}{n}\right)^n$$

Therefore $\lim_{n\to\infty} \phi_{B(n,\frac{\lambda}{n})}(t) = \exp(\lambda e^{it} - \lambda)$

This is as same as $\phi_{P(\lambda)}(t)$.

By the Continuity theorem, the binomial distribution with parameter n and $\frac{\lambda}{n}$ converges to the poisson distribution with parameter λ as $n \to \infty$ in distribution.

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