Probability on a game

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TRPG is a kind of table game that is played with some dice. Here, I introduce a probability in one of TRPG

In the game "Double cross", we use 10-sided dice to decide a number in this way below.

① Some definition.
- C: Critical number ($C \in \{2, 3, 4, \ldots, 10\}$)
- t: discrete time ($t = 0, 1, 2, \ldots$)
- N_t: number of dice to roll (the first number of dice is N_0)

② Roll N_t 10-sided fair dice

③ Remove all dice that result is smaller than C. (Keep others.)

④ Decide N_{t+1} as number of remaining dice.

⑤ Add 1 to t and repeat ②~⑤

For example, if $C = 8$,

$t=0$ $N_0 = 10$ Roll(at $t=0$) $\Rightarrow \{\underbrace{2, 2, 3, 5, 5, 6, 7}_{\text{Removed}}, \underbrace{8, 8, 9}_{\text{remain}}\}$ $\rightarrow N_1 = 3$

$t=1$ $N_1 = 3$ Roll(at $t=1$) $\Rightarrow \{\underbrace{4}_{\text{Removed}}, \underbrace{8, 10}_{\text{remain}}\}$ $\rightarrow N_2 = 2$

$t=2$ $N_2 = 2$ Roll(at $t=2$) $\Rightarrow \{\underbrace{2, 5}_{\text{Removed}}\}$ $\rightarrow N_3 = 0$

t_c is decided by last time t that $N_t \neq 0$.
In the example, critical happened 2 times and $t_c = 2$.

As observations,
- $N_t \leq N_s$ if $t \geq s$ (decreasing)
- This is discrete time and no space process, so it is similar to blanching process
- If $N_0 = 1$, it is very simple. (

$$P(N_1 = 1 \mid N_0 = 1) = \frac{11-C}{10} =: u, \quad P(N_1 = 0 \mid N_0 = 1) = \frac{C-1}{10} \; (= 1-u)$$

This means $N_1 \sim \text{Bernoulli}(u)$

- One time step is also easy to time think about. Comparing time t and $t+1$, with $N_t = n$

$$P(N_{t+1} = k \mid N_t = n) = \binom{n}{k} u^k (1-u)^{n-k}$$

This means $N_{t+1} \sim \text{Binomial}(N_t, u)$

Here, we set 3 series of probabilities.

p is about t_c

$P(t_c) = \mathbb{P}\{$ Critical happens t_c times $\}$

$= \mathbb{P}\{$ N become 0 for the first time at time $t_c+1 \}$

$= \mathbb{P}(N_{t_c+1} = 0 \sqcup N_{t_c} \neq 0)$

$= \mathbb{P}(N_{t_c+1} = 0 \mid N_{t_c} \neq 0)\, \mathbb{P}(N_{t_c} = 0)$

q is sum of p.

$q(t) = P(0) + P(1) + P(2) + \dots\ P(t)$

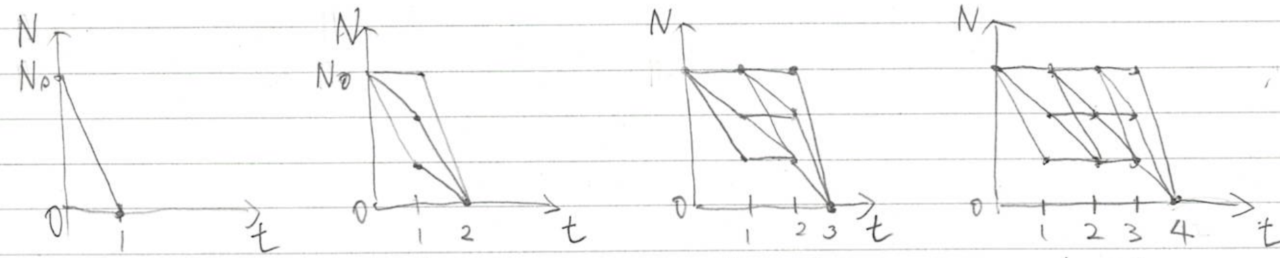
$= \mathbb{P}(N_1 = 0 \sqcup N_0 \neq 0) + \mathbb{P}(N_2 = 0 \sqcup N_1 \neq 0) + \mathbb{P}(N_3 = 0 \sqcup N_2 \neq 0) + \dots + \mathbb{P}(N_{t+1} = 0 \sqcup N_t \neq 0)$

$= \mathbb{P}(N_{t+1} = 0)$

r is about single process with numbers of dice before and after theprocess.

$r(a,b) = \mathbb{P}(N_{t+1} = b \mid N_t = a \quad \text{for } 0 \le b \le a)$

$= \binom{a}{b} u^b (1-u)^{a-b}$



$t_c = 0$ with $P(0)$ | $t_c = 1$ with $P(1)$ | $t_c = 2$ with $P(2)$ | $t_c = 3$ with $P(3)$

In this report, I tried to compute P, q, and $\mathbb{E}(t_c)$.

set $N_0 = n$.

$P(0) = \mathbb{P}(N_1 = 0 \mid N_0 \neq 0)\, \mathbb{P}(N \neq 0)$

$= \mathbb{P}(N_1 = 0 \mid N_0 = n)\, \mathbb{P}(N_0 = n)$

$= r(n,0) \cdot 1$

$P(1) = \mathbb{P}(N_2 = 0 \mid N_1 \neq 0)\, \mathbb{P}(N_1 \neq 0)$

$= \sum_{k=1}^{n} \mathbb{P}(N_2 = 0 \mid N_1 = k)\, \mathbb{P}(N_1 = k)$ ($\because \{N_1 \neq 0\} = \{N_1 = \{1, 2, \dots n\}\}$ and partition theorem)

$= \sum_{k=1}^{n} r(k,0)\, \mathbb{P}(N_1 = k \mid N_0 = n)$ ($\because \{N_0 = n\}$ always happens)

$= \sum_{k=1}^{n} r(k,0)\, r(n,k)$

$= \sum_{k=0}^{n} r(n,k)\, r(k,0) - r(n,0) \cdot r(0,0) = \sum_{k=0}^{n} r(n,k)\, r(k,0) - P(0)$

$$P(2) = \mathbb{P}(N_3=0 \mid N_2 \neq 0)\ \mathbb{P}(N_2 \neq 0)$$
$$= \sum_{k=1}^{n} \mathbb{P}(N_3=0 \mid N_2=k)\ \mathbb{P}(N_2=k)$$
$$= \sum_{k=1}^{n} r(k,0)\ \mathbb{P}(N_2=k \mid N_1 \geq k)\ \mathbb{P}(N_1 \geq k) \qquad (\because \{N_2=k\} \subset \{N_1 \geq k\})$$
$$= \sum_{k=1}^{n} r(k,0) \sum_{l=k}^{n} \mathbb{P}(N_2=k \mid N_1=l)\ \mathbb{P}(N_1=l)$$
$$= \sum_{k=1}^{n} \sum_{l=k}^{n} r(k,0)\, r(l,k)\ \mathbb{P}(N_1=l \mid N_0=n)\ \mathbb{P}(N_0=n)$$
$$= \sum_{k=1}^{n} \sum_{l=k}^{n} r(n,l)\, r(l,k)\, r(k,0)$$
$$= \sum_{k=0}^{n} \sum_{l=k}^{n} r(n,l)\, r(l,k)\, r(k,0) - \sum_{l=1}^{n} r(n,l)\, r(l,0)\, r(0,0) - r(n,0)\, r(0,0)\, r(0,0)$$
$$= \sum_{k=0}^{n} \sum_{l=k}^{n} r(n,l)\, r(l,k)\, r(k,0) - P(0) - P(1)$$

On the analogy of this, by using $\{N_t\}_{t=0}^{t_c+1}$ such that
$N_0 = n$, $N_{t_c+1} = 0$, $N_{t+1} \leq N_t$,

$$q(t_c) = \mathbb{P}(N_{t_c+1}=0)$$
$$= \sum_{N_{t_c}=0}^{n} \sum_{N_{t_c-1}=N_{t_c}}^{n} \cdots \sum_{N_1=N_2}^{n} r(n,N_1)\, r(N_1,N_2) \cdots r(N_{t_c},0)$$

(to simplify this, summations is denoted by $\sum_{\{N_t\}}$)

$$= \sum_{\{N_t\}} \binom{n}{N_1} u^{N_1}(1-u)^{n-N_1} \binom{N_1}{N_2} u^{N_2}(1-u)^{n-N_2} \cdots \binom{N_{t_c}}{0} u^{0}(1-u)^{N_{t_c}-0}$$
$$= \sum_{\{N_t\}} \binom{n}{N_1}\binom{N_1}{N_2} \cdots \binom{N_{t_c}}{0} u^{N_1+N_2+\cdots+N_{t_c}+0}\ (1-u)^{n-N_1+N_1-N_2+\cdots+N_{t_c-1}-N_{t_c}+N_{t_c}-0}$$
$$= (1-u)^n \sum_{\{N_t\}} u^{\sum_{t=1}^{t_c+1} N_t}\ \frac{n!}{N_1!\,(n-N_1)!}\ \frac{N_1!}{N_2!\,(N_1-N_2)!} \cdots \frac{N_{t_c-1}!}{N_{t_c}!\,(N_{t_c-1}-N_{t_c})!}\ \frac{N_{t_c}!}{0!\,N_{t_c}!}$$
$$= (1-u)^n\, n! \sum_{\{N_t\}} u^{\sum_{t_c}} \frac{1}{(n-N_1)!\,(N_2-N_1)! \cdots (N_{t_c-1}-N_{t_c})!\,N_{t_c}!}$$

Since elements of summation are too much, it cannot be computed more.

On the other hand, this problem is transfered like this.

$N_1 \sim \mathrm{Binomial}(N_0, u)$
$N_2 \sim \mathrm{Binomial}(N_1, u)$
$\vdots$
$N_t \sim \mathrm{Binomial}(N_{t-1}, u)$

From this, average number of dice at time t can be

computed.
For $X \sim B(n, p)$, $\mathbb{E}(X) = np$, so $\mathbb{E}(N_1) = N_0 u$ and from iteration, $\mathbb{E}(N_t) = N_0 u^t$. Number of dice is decreasing exponentially.
However, even from this method, $\mathbb{E}(t_c)$ is hard to compute