Introduction to probability

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Exercise : Sum of two independent random variables that follow normal distribution.

(1)

Proof of that if X, Y are independent and follow $N(0,1)$ then $X+Y \sim N(0,2)$

Let X, Y be random variables :
X and Y are independent.
X and Y follow $N(0,1)$,
f_{XY} is joint density function and f_X, f_Y are probability density function of X, Y.

$Z := X+Y$ is also a random variable. For fixed z,

$$\mathbb{P}(Z \leq z) = \mathbb{P}(X+Y \leq z)$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\right)dx$$

By substituting y with $y' = y+x$,
($y = z-x \Leftrightarrow y' = z$, $y = -\infty \Leftrightarrow y' = -\infty$, $dy' = dy$),

$$\mathbb{P}(Z \leq z) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z'} f_{XY}(x, y'-x)\,dy'\right)dx$$

$$= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} f_{XY}(x, y-x)\,dx\right)dy$$

So, probability density function $f_Z(y) = \int_{-\infty}^{\infty} f_{XY}(x, y-x)\,dx$.
Since X and Y are independent, $f_{XY}(x, y-x) = f_X(x)\, f_Y(y-x)$.

From definition of normal destribution,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad f_Y(y-x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2}\right)$$

$$f_Z(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2}\right) dx.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2+x^2-2xy+y^2}{2}\right) dx.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\left(x-\frac{1}{2}y\right)^2 - \frac{y^2}{2} + \frac{1}{4}y^2\right) dx$$

$$= \frac{1}{2\pi} \exp\left(-\frac{y^2}{4}\right) \int_{-\infty}^{\infty} \exp\left(-\left(x-\frac{1}{2}y\right)^2\right) dx.$$

By substituting x with $x' = x - \frac{1}{2}y$
$(x = -\infty \leftrightarrow x' = -\infty,\ x = \infty \leftrightarrow x' = -\infty,\ dx' = dx)$

$$f_Z(y) = \frac{1}{2\pi}\exp\left(-\frac{y^2}{4}\right)\int_{-\infty}^{\infty}\exp(-x^2)\,dx$$

With using Gaussian integral $\int_{-\infty}^{\infty}\exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}}$,

$$f_Z(y) = \frac{1}{2\pi}\exp\left(-\frac{y^2}{4}\right)\cdot\sqrt{\frac{\pi}{1}}$$

$$= \frac{1}{\sqrt{2\pi\cdot 2}}\exp\left(-\frac{y^2}{2\cdot 2}\right)$$

Therefore, $Z \sim N(0, 2)$ □ //

(2) General caces.

Most of the conditions are the same with (1). However, X follows $N(0, \sigma^2)$ and Y follows $N(0, \tau^2)$.

Thus, $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$, $f_Y(y-x) = \frac{1}{\sqrt{2\pi\tau^2}}\exp\left(-\frac{(y-x)^2}{2\tau^2}\right)$,

$$f_Z(y) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)\cdot\frac{1}{\sqrt{2\pi\tau^2}}\exp\left(-\frac{(y-x)^2}{2\tau^2}\right)dx$$

$$= \frac{1}{\sqrt{4\pi^2\sigma^2\tau^2}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2\sigma^2} - \frac{(y-x)^2}{2\tau^2}\right)dx$$

$$= \frac{1}{\sqrt{4\pi^2\sigma^2\tau^2}}\int_{-\infty}^{\infty}\exp\left(-\frac{\tau^2x^2 + \sigma^2x^2 - 2\sigma^2xy + \sigma^2y^2}{2\sigma^2\tau^2}\right)dx$$

$$= \frac{1}{\sqrt{4\pi^2\sigma^2\tau^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{\tau^2+\sigma^2}{2\sigma^2\tau^2}\left(x - \frac{\sigma^2}{\tau^2+\sigma^2}y\right)^2 + \frac{\tau^2+\sigma^2}{2\sigma^2\tau^2}\cdot\left(\frac{\sigma^2}{\tau^2+\sigma^2}\right)^2y^2 - \frac{\sigma^2}{2\sigma^2\tau^2}y^2\right\}dx$$

$$= \frac{1}{\sqrt{4\pi^2\sigma^2\tau^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{\tau^2+\sigma^2}{2\sigma^2\tau^2}\left(x - \frac{\sigma^2}{\tau^2+\sigma^2}y\right)^2 + \frac{1}{2\sigma^2\tau^2}\left(\frac{\sigma^4}{\tau^2+\sigma^2} - \sigma^2\right)y^2\right\}dx$$

$$= \frac{1}{\sqrt{4\pi^2\sigma^2\tau^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{\tau^2+\sigma^2}{2\sigma^2\tau^2}\left(x - \frac{\sigma^2}{\tau^2+\sigma^2}y\right)^2\right\}\exp\left\{\frac{1}{2\sigma^2\tau^2}\left(\frac{\sigma^4 - \sigma^2\tau^2 - \sigma^4}{\tau^2+\sigma^2}\right)y^2\right\}dx$$

$$= \frac{1}{\sqrt{4\pi^2\sigma^2\tau^2}}\exp\left(-\frac{y^2}{2(\tau^2+\sigma^2)}\right)\int_{-\infty}^{\infty}\exp\left\{-\frac{\tau^2+\sigma^2}{2\sigma^2\tau^2}\left(x - \frac{\sigma^2}{\tau^2+\sigma^2}y\right)^2\right\}dx.$$

After substituting x with $x' = x - \frac{\sigma^2}{\tau^2+\sigma^2}y$, by using Gaussian integration $\int_{-\infty}^{\infty}\exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}}$

$$f_Z(y) = \frac{1}{\sqrt{4\pi^2\sigma^2\tau^2}}\exp\left(-\frac{y^2}{2(\tau^2+\sigma^2)}\right)\sqrt{\pi\cdot\frac{2\sigma^2\tau^2}{\tau^2+\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\cdot(\tau^2+\sigma^2)}}\exp\left(-\frac{y^2}{2(\tau^2+\sigma^2)}\right)$$

So $Z \sim N(0, \tau^2 + \sigma^2)$.

For 2 independent random variable $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \tau^2)$
$X + Y$ follows $N(0, \sigma^2 + \tau^2)$ $\square$

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