Exercise for The Distribution Function

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Question

Show that the density function

$$f(x)=\begin{cases} \frac{1}{\pi\sqrt{x(1-x)}} & 0<x<1, \\ 0 & otherwise, \end{cases}$$

has the distribution with the form

$$F(x)=c\sin^{-1}(\sqrt{x}),\ 0<x<1,$$

and find the constant c.

Solution

Set $u=\sqrt{x}$, then

$$\begin{aligned} \mathrm{d}u=&\frac{1}{2\sqrt{x}}\ \mathrm{d}x \\ \mathrm{d}x=&2\sqrt{x}\ \mathrm{d}u \end{aligned}$$

Then substitute the expression of u into the equation below, we calculate the distribution of the probability density function:

$$\begin{aligned} \int_{-\infty}^{t} f(x)\mathrm{d}x=&\int_{0}^{t}\frac{1}{\pi\sqrt{x\,(1-x)}}\ \mathrm{d}x \\ =&\int_{0}^{\sqrt{t}}\frac{2\sqrt{x}}{\pi\sqrt{x\,(1-x)}}\ \mathrm{d}u \\ =&\int_{0}^{\sqrt{t}}\frac{2}{\pi\sqrt{1-u^2}}\ \mathrm{d}u \\ =&\frac{2}{\pi}\int_{0}^{\sqrt{t}}\frac{1}{\sqrt{1-u^2}}\ \mathrm{d}u,\ 0<u<1. \end{aligned}$$

Therefore, we can obtain the distribution:

$$F(x)=\frac{2}{\pi}\sin^{-1}\sqrt{x},\ 0<x<1,$$

and $c:=\frac{2}{\pi}$. $\square$