Exercise for Dice Problem

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Question

Two fair dice are thrown. Let A be the event that the first shows an odd number, B be the event that the second shows an even number, and C be the event that either both are odd or both are even. Show that A, B, C are pairwise independent but not independent.

Proof

From the problem, we can see the 3 events A, B, C $\in (\Omega, \mathcal{F}, \mathbb{P})$ such that:

- A: First dice being odd,
- B: Second dice being even,
- C: Both being odd/even.

All the events are shown below:

First	Second
Even	Even
Even	Odd
Odd	Even
Odd	Odd

First, we need to show that A, B, C are pairwise independent:
For A and B:
According to Definition 1.38, if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, A and B are independent.
In this case, A and B represent the events of throwing two identical dices, so that their results do not effect each other.
$\Longrightarrow$ A and B are independent to each other.

For A and C, B and C:
According to (1.37), events A and B are independent, if $\mathbb{P}(A|B) = \mathbb{P}(A)$ and $\mathbb{P}(B|A) = \mathbb{P}(B)$ when $\mathbb{P}(A), \mathbb{P}(B) > 0$. And from Definition 1.31, if $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$, A and B can be proved as independent events to each other.
Then we can be informed from the graph above that

$$\mathbb{P}(A|C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{1/4}{1/2} = \frac{1}{2} = \mathbb{P}(A),$$
$$\mathbb{P}(C|A) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(A)} = \frac{1/4}{1/2} = \frac{1}{2} = \mathbb{P}(C),$$

Also,

$$\mathbb{P}(B|C) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} = \frac{1/4}{1/2} = \frac{1}{2} = \mathbb{P}(B),$$
$$\mathbb{P}(C|B) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(B)} = \frac{1/4}{1/2} = \frac{1}{2} = \mathbb{P}(C).$$

$\Longrightarrow$ A and C, B and C are independent to each other.
Therefore, we can prove that A, B and C are pairwise independent.

Then, we need to show that A, B, C are not independent:
Based on the Problem, $\mathbb{P}(C) = \mathbb{P}(\text{both odd}) + \mathbb{P}(\text{both even})$

$$\begin{aligned}\Longrightarrow \mathbb{P}(C) =& \mathbb{P}(A)(1-\mathbb{P}(B)) + \mathbb{P}(B)(1-\mathbb{P}(A)) \\ =& \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A)\mathbb{P}(B).\end{aligned}$$

To calculate $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ substituting the previous equation:

$$\begin{aligned}\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) =& \mathbb{P}(A)\mathbb{P}(B)[\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A)\mathbb{P}(B)] \\ =& \mathbb{P}(A)^2\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B)^2 - 2[\mathbb{P}(A)\mathbb{P}(B)]^2 \\ =& \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} - 2\cdot(\frac{1}{4})^2 \\ =& \frac{1}{8}.\end{aligned}$$

However, to determine the value of $\mathbb{P}(A \cap B \cap C)$: Since A and B can not be satisfied at the same time under the situation of both dices being even or odd, the probability of the intersection of A, B and C are 0.
Therefore, $\mathbb{P}(A \cap B \cap C) = 0$.
$\Longrightarrow \mathbb{P}(A \cap B \cap C) \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$
$\Longrightarrow$ A, B, C are not independent.

Above all, A, B, and C are pairwise independent but not independent. □