There are n socks in a drawer, three of which are red and the rest black. John chooses his socks by selecting two at random from the drawer, and puts them on. He is three times more likely to wear socks of different colours than to wear matching red socks. Find n.
For this value of n, what is the probability that John wears matching black socks? (Cambridge 2008)

Let n be the number of socks

There are 3 red socks and $(n-3)$ black socks.

A_i: choosing red sock

B_i: choosing black sock. $\quad i = 1, 2$ (order of choosing)

- Taking red sock first: $\mathbb{P}(A_1) = \frac{3}{n}$
  - taking black sock second: $\mathbb{P}(B_2 | A_1) = \frac{n-3}{n-1}$ (there are $n-3$ black socks in $n-1$ total)
  - taking red sock second: $\mathbb{P}(A_2 | A_1) = \frac{2}{n-1}$ (there are 2 red socks left in $n-1$ total)
- Taking black sock first: $\mathbb{P}(B_1) = \frac{n-3}{n}$
  - taking red sock second: $\mathbb{P}(A_2|B_1) = \frac{3}{n-1}$ (there are 3 red socks in $n-1$ total)
  - taking black sock second: $\mathbb{P}(B_2|B_1) = \frac{n-4}{n-1}$ (there are $n-4$ black socks left in $n-1$ total)

Probability taking 2 different color socks:

$$\mathbb{P}(A\&B) = \mathbb{P}(A_2 \cap B_1) + \mathbb{P}(B_2 \cap A_1) = \mathbb{P}(B_2|A_1)\mathbb{P}(A_1) + \mathbb{P}(A_2|B_1)\mathbb{P}(B_1)$$
$$= \frac{2.3(n-3)}{n(n-1)}$$

Probability taking 2 red:

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1).\mathbb{P}(A_2|A_1) = \frac{2.3}{n(n-1)}$$

Since $\frac{\mathbb{P}(A\&B)}{\mathbb{P}(A_1 \cap A_2)} = 3 \Rightarrow \frac{2.3(n-3)}{2.3} = 3 \Rightarrow \boxed{n = 6}$

Probability John wears matching black socks:

$$\mathbb{P}(B_1 \cap B_2) = \mathbb{P}(B_2|B_1)\,\mathbb{P}(B_1) = \frac{(n-4)(n-3)}{n(n-1)} = \frac{2.3}{6.5} = \frac{1}{5}$$