Fair dice thrown 12000 times

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This report solves Exercise 8.32 of the textbook, *Probability, an introduction.*

Exercise 8.32
A fair dice is thrown 12,000 times. Use the central limit theorem to find values of a and b such that

$$\mathbb{P}(1900 < S < 2200) \approx \int_a^b \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy, \tag{1}$$

where S is the total number of sixes thrown.
Solution
Denote the number of times the fair dice is thrown by n.
As in *Alternative form of Chebyshev's inequality and rolling of a fair dice* by Lin Guozhang, for $i = 1, ..., n$, define Z_i as follows.

$$Z_i = \begin{cases} 1, & \text{if the dice gives 6 for the i-th roll.} \\ 0, & \text{otherwise.} \end{cases}$$

For a fair dice, $Z_1, ..., Z_n$ are independent, and $\mathbb{P}(Z_i = 1) = \frac{1}{6}$ for $i = 1, ..., n$.
Thus, $Z_i \sim B(\frac{1}{6})$ for $i = 1, ..., n$, with $\mu = \frac{1}{6}$ and $\sigma = \frac{\sqrt{5}}{6}$.
By central limit theorem, set

$$\begin{aligned} Y_n &:= \frac{\sum_{i=1}^n Z_i - n\mu}{\sigma\sqrt{n}} \\ &= \frac{\sum_{i=1}^n Z_i - \frac{1}{6}n}{\frac{\sqrt{5}}{6}\sqrt{n}} \end{aligned} \tag{2}$$

Then, for any $x \in \mathbb{R}$, as $n \to \infty$,

$$\mathbb{P}(Y_n \leq x) \to \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy$$

In this exercise, n=12000, a very large number.
Therefore,

$$\mathbb{P}(Y_{12000} \leq x) \approx \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy \tag{3}$$

By (1),

$$Y_{12000} = \frac{\sum_{i=1}^{12000} Z_i - \frac{1}{6}12000}{\frac{\sqrt{5}}{6}\sqrt{12000}}$$

According to the definition given in the problem, $S = \sum_{i=1}^{12000} Z_i$.
Thus,

$$Y_{12000} = \frac{S - 2000}{\frac{50}{3}\sqrt{6}} \tag{4}$$

$$\mathbb{P}(\frac{S - 2000}{\frac{50}{3}\sqrt{6}} \leq x) = \mathbb{P}(S \leq \frac{50}{3}\sqrt{6}x + 2000) \tag{5}$$

By (3) to (5),

$$\mathbb{P}(S \leq \frac{50}{3}\sqrt{6}x + 2000) \approx \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy$$

Let $\frac{50}{3}\sqrt{6}x + 2000 = 1900$. Then $x = -\sqrt{6}$.
Let $\frac{50}{3}\sqrt{6}x + 2000 = 2200$. Then $x = 2\sqrt{6}$. Thus,

$$\mathbb{P}(S \leq 1900) \approx \int_{-\infty}^{-\sqrt{6}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy$$
$$\mathbb{P}(S \leq 2200) \approx \int_{-\infty}^{2\sqrt{6}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy$$

$$\begin{aligned}\mathbb{P}(1900 < S < 2200) &\approx \mathbb{P}(S \leq 2200) - \mathbb{P}(S \leq 1900) \\ &= \int_{-\sqrt{6}}^{2\sqrt{6}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy \end{aligned} \tag{6}$$

Therefore, $a = -\sqrt{6}$ and $b = 2\sqrt{6}$.

The following introduces Monte Carlo integration to calculate (6) numerically.
The mean of a function over the interval (a, b) is

$$< f(y) >= \frac{1}{b-a} \int_a^b f(y)dy. \tag{7}$$

By Law of Large numbers,

$$< f(y) >\approx \frac{1}{N} \sum_{i=1}^{N} f(y_i), \tag{8}$$

where N is sufficiently large, $Y_i \sim U(a, b)$ and independent.
By (7) and (8),

$$\int_a^b f(y)dy \approx (b-a)\frac{1}{N} \sum_{i=1}^{N} f(y_i). \tag{9}$$

In (6), $f(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)$, $a = -\sqrt{6}$,and $b = 2\sqrt{6}$. Set N=1000000.
Then, the following python code computes (6) numerically.

```
from scipy import random
import numpy as np

#Set up according to (6)
def f(y):
    return 1/(np.sqrt(2*np.pi))*np.exp(-np.power(y,2)/2)

a = -np.sqrt(6)
b = 2*np.sqrt(6)
N = 1000000

#To calculate the sum on RHS of (9), introduce the variable, sum.
sum = 0.0

#Use a loop to calculate the sum
for i in range(N):
    #Y_i obeys U(a,b) and is independent of the others.
    y_i = random.uniform(a,b)
    sum += f(y_i)

int_value = (b-a)/float(N)*sum

print ("The value calculated by Monte Carlo integration is {}.".format(int_value))
```

Since the value of y_i is random, the value on RHS of (9) is random.
However, the result is always very close to 0.992.
One typical result (random seed set to be 1) is the following.

```
The value calculated by Monte Carlo integration is 0.99225035603056.
```

Therefore, by Monte Carlo integration,

$$\mathbb{P}(1900 < S < 2200) \approx \int_{-\sqrt{6}}^{2\sqrt{6}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)dy \approx 0.992.$$

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- *Alternative form of Chebyshev's inequality and rolling of a fair dice* by Lin Guozhang
- `https://www.geeksforgeeks.org/monte-carlo-integration-in-python/`
- `https://en.wikipedia.org/wiki/Mean_of_a_function`