Gamma Function and Gamma Distribution

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This report explores the gamma function and the gamma distribution. It is related to Exercise 5.45 to 5.47 of the textbook, *Probability: an introduction.*

According to page 69 of the textbook, the gamma distribution and the gamma function are defined as follows. The gamma distribution with parameters $w(>0)$ and $\lambda(>0)$ has density function

$$f(x) = \begin{cases} \frac{1}{\Gamma(w)}\lambda^w x^{w-1} e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{1.1}$$

where $\Gamma(w)$ is the gamma function, defined by

$$\Gamma(w) = \int_0^\infty x^{w-1} e^{-x} dx \tag{1.2}$$

In this report, only non-negative real w is considered, since only such w is important for the gamma distribution, as its density function suggests.

Properties of the gamma function

$$\Gamma(1/2) = \sqrt{\pi} \tag{2.1}$$

$$\Gamma(w+1) = w\Gamma(w) \tag{2.2}$$

$$\Gamma(w+1) = w!, \text{ for } w \in \mathbb{N} \tag{2.3}$$

Proof of (2.1)

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx$$

Let $u := x^{1/2}$.

$$\Gamma(1/2) = \int_0^\infty 2e^{-u^2} du$$

Since the integrand is an even function, the integral can be rewritten as

$$\Gamma(1/2) = \int_{-\infty}^\infty e^{-u^2} du$$

$$(\Gamma(1/2))^2 = \int_{-\infty}^\infty e^{-u^2} du \int_{-\infty}^\infty e^{-u^2} du$$

Since u is a dummy variable, it can be replaced with v.

$$\begin{aligned}(\Gamma(1/2))^2 &= \int_{-\infty}^\infty e^{-u^2} du \int_{-\infty}^\infty e^{-v^2} dv \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(u^2+v^2)} du dv\end{aligned}$$

Integrate in polar coordinates, where $\rho^2 := x^2 + y^2$, $\phi = tan^{-1}(y/x)$.

$$
\begin{aligned}
(\Gamma(1/2))^2 &= \int_0^\infty \int_0^{2\pi} e^{-\rho^2} \rho d\phi d\rho \\
&= 2\pi \int_0^\infty e^{-\rho^2} \rho d\rho
\end{aligned}
$$

Let $s := -\rho^2$.

$$
\begin{aligned}
(\Gamma(1/2))^2 &= -\pi \int_0^{-\infty} e^s ds \\
&= \pi
\end{aligned}
$$

Thus,

$$\Gamma(1/2) = \sqrt{\pi}$$

Q.E.D.

Proof of (2.2)

According to (1.2),

$$\Gamma(w+1) = \int_0^\infty x^w e^{-x} dx$$

By integration by parts,

$$
\begin{aligned}
\Gamma(w+1) &= -x^w e^{-x}|_0^\infty + \int_0^\infty w x^{w-1} e^{-x} dx \\
&= 0 + \int_0^\infty w x^{w-1} e^{-x} dx \\
&= w\Gamma(w)
\end{aligned}
$$

Q.E.D.

Proof of (2.3)
Proof by induction
Base case: $w = 0$

$$
\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-x} dx \\
&= 1 \\
&= 0!
\end{aligned}
$$

Induction step:
Assume $\Gamma(k+1) = k!$, for $k \in \mathbb{N}$.
Then, with the assumption and (2.2),

$$
\begin{aligned}
\Gamma(k+2) &= (k+1)\Gamma(k+1) \\
&= (k+1)k! \\
&= (k+1)!
\end{aligned}
$$

From base case and induction step, (2.3) is proved.

Q.E.D.

Properties of the gamma distribution

1. If $X_1, ..., X_n$ are independent, and $X_i \sim \Gamma(w_i, \lambda)$, then
$$Y := \Sigma_{i=1}^n X_i \sim \Gamma(\Sigma_{i=1}^n w_i, \lambda)$$
2. If $X \sim \Gamma(w, \lambda)$, then $aX \sim \Gamma(w, \lambda/a)$ for $a > 0$.

Sketch of proof of property 1

The proof of property 1 above can be based on Matsumoto Kosuke and Nguyen Duc Thanh's report, *On the Gamma Distribution*, where it is proved that if X and Y are independent, and $X \sim \Gamma(s, \lambda)$ and $Y \sim \Gamma(t, \lambda)$, then

$$X + Y \sim \Gamma(s + t, \lambda)$$

Then, with the use of mathematical induction, property 1 can be proved.

Proof of property 2

As proved in *On the Gamma Distribution*, the moment generating function of a random variable $X \sim \Gamma(w, \lambda)$ is

$$M_X(t) = (\frac{\lambda}{\lambda - t})^w, \text{ for } |t| < \lambda.$$

Then, for $|t| < \lambda/a$,

$$\begin{aligned} M_{aX}(t) &= E(e^{t(aX)}) \\ &= E(e^{at(X)}) \\ &= M_X(at) \\ &= (\frac{\lambda}{\lambda - at})^w \\ &= (\frac{\lambda/a}{\lambda/a - t})^w \end{aligned}$$

By the uniqueness of moment generating function, $aX \sim \Gamma(w, \lambda/a)$.

Q.E.D.

Remark

The gamma distribution is a generalization of the exponential distribution, that is,

$$\Gamma(1, \lambda) = E(\lambda)$$

Indeed, if $w = 1$ is substituted in (1.1), then the density function becomes the same as the one for the exponential distribution.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- `https://en.wikipedia.org/wiki/Gaussian_integral#By_polar_coordinates`
- *On the Gamma Distribution* by Matsumoto Kosuke and Nguyen Duc Thanh. (`http://www.math.nagoya-u.ac.jp/~richard/teaching/s2021/SML_Matsumoto_Tom.pdf`)